-27(4): Relativistic Quantum Theory

In special relativity and in ECE2 theory, the hamiltonian is:

$$H = E + U \quad -(1)$$

where $E$ is the total relativistic energy:

$$E = \gamma m c^2 = \left(c^2 p^2 + m^2 c^4\right)^{1/2} \quad -(2)$$

in which the Lorentz factor is

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(3)$$

and where $v_N$ is the Newtonian linear velocity. Here $m$ is the mass of a particle subjected to the potential energy $U$, and $c$ is the speed of light. In eq. (2) $p$ is the relativistic momentum:

$$p = \gamma m v_N . \quad -(4)$$

The quantization of these equations was first considered by Sommerfeld in 1913 / 1914 and about ten years later by Dirac. In the UFT papers the quantization procedure was developed in many directions, giving several new types of spectroscopy and deriving the fermion equation from Cartan geometry.

Consider:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(5)$$

and write it as:

$$(E - mc^2)(E + mc^2) = c^2 p^2 \quad -(6)$$

It follows that:

$$E - mc^2 = \frac{c^2 p^2}{E + mc^2} \quad -(7)$$

So:

$$E = \frac{c^2 p^2}{E + mc^2} + mc^2 \quad -(8)$$

From eqs. (1) and (8):

$$H - U - mc^2 = \frac{c^2 p^2}{H - U + mc^2} \quad -(9)$$

where

$$H = \gamma mc^2 + U. \quad -(10)$$

Dirac made the approximations:

$$U << E \quad -(11)$$

and

$$H \sim E \sim mc^2 \quad -(12)$$

so eq. (9) becomes:

$$H = \frac{c^2 p^2}{2mc^2 - U} + mc^2 + U \quad -(13)$$

$$= \frac{p^2}{2m}\left(1 - \frac{U}{2mc^2}\right)^{-1} + mc^2 + U \quad -(14)$$

Assuming that:

$$U << 2mc^2 \quad -(15)$$

it follows that:

$$H \sim \frac{p^2}{2m}\left(1 + \frac{U}{2mc^2}\right) + mc^2 + U \quad -(16)$$

At this point Dirac introduced the SU(2) basis described in many UFT papers and books, and in many volumes in the literature. Therefore:

$$H \sim \frac{1}{2m} \underline{\sigma} \cdot \underline{p} \left(1 + \frac{U}{2mc^2}\right) \underline{\sigma} \cdot \underline{p} + mc^2 + U \quad -(17)$$

In the presence of a magnetic field:

$$\underline{p} \rightarrow \underline{p} - e\underline{A} \quad -(18)$$

where $\underline{A}$ is the vector potential, so:

$$H = \frac{1}{2m}\,\underline{\sigma}\cdot(\underline{p}-e\underline{A})\left(1+\frac{U}{2mc^2}\right)\underline{\sigma}\cdot(\underline{p}-e\underline{A}) + mc^2 + U \quad -(19)$$

The theory is now quantized with Schroedinger's rule:

$$\underline{p}\,\phi = -i\hbar\,\underline{\nabla}\phi \quad -(20)$$

The above is a famous and very successful theory which describes many phenomena in one equation (19). Nevertheless it contains weak points that have been covered in the UFT series of papers. The theory gives the half integral spin of the electron, the Landé factor of the electron, an electron g factor of two, and the fine structure of atoms and molecules. It also gives the Darwin effect. During the course of development of the UFT series many other effects have been discovered.

The extension of this theory to m theory should therefore produce many new effects, notably effects in the fine structure of atoms and molecules. In m theory the hamiltonian (1) becomes:

$$H = m(r_1)\,\gamma mc^2 + U \quad -(21)$$

where:

$$\gamma = \left(m(r_1) - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(22)$$

in a basis defined by:

$$r_1 = \frac{r}{m(r)^{1/2}} \quad -(23)$$

and

$$v_{1N} = \frac{v_N}{m(r)^{1/2}} \quad -(24)$$

In the theory, eq. (5) becomes:

$$E^2 = m(r_1)\left(c^2 p_1^2 + m^2 c^4\right) \quad -(25)$$

so

$$E^2 - m(r_1) m^2 c^4 = m(r_1) c^2 p_1^2 \quad -(26)$$

i.e.

$$\left(E - m(r_1)^{1/2} mc^2\right)\left(E + m(r_1)^{1/2} mc^2\right) = m(r_1) c^2 p_1^2 \quad -(27)$$

so

$$E = H - U = \frac{m(r_1) c^2 p_1^2}{E + m(r_1)^{1/2} mc^2} + m(r_1)^{1/2} mc^2 \quad -(28)$$

and:

$$H = \frac{m(r_1) c^2 p_1^2}{H - U + m(r_1)^{1/2} mc^2} + m(r_1)^{1/2} mc^2 + U \quad -(29)$$

Now apply the Dirac type approximations to eqn. (29):

$$U << E \quad -(30)$$

so

$$H \sim E = m(r_1) \gamma mc^2 \quad -(31)$$

From eqs (22) and (31):

$$H \sim m(r_1)^{3/2}\left(1 - \frac{v_{1N}^2}{c^2}\right)^{-1/2} mc^2 \quad -(32)$$

$$\rightarrow m(r_1)^{3/2} mc^2 \quad -(33)$$

if

$$v_{IN} << c \quad -(34)$$

Therefore:

$$H = \frac{m(r_1) c^2 p_1^2}{\left(m^{3/2}(r_1) + m^{1/2}(r_1)\right) mc^2 - U} + m(r_1)^{1/2} mc^2 + U \quad -(35)$$

$$= \frac{c^2 p^2}{m^{1/2}(r)\left(1 + m(r)\right) mc^2 - U} + m(r)^{1/2} mc^2 + U$$

using

$$p_1^2 = \frac{p^2}{m(r)} \quad -(36)$$

Therefore:

$$H = \frac{p^2}{m^{1/2}(r)\left(1+m(r)\right) m}\left(1 - \frac{U}{m^{1/2}(r)\left(1+m(r)\right) mc^2}\right)^{-1} + m(r)^{1/2} mc^2 + U \quad -(37)$$

If

$$U << mc^2 \quad -(38)$$

ter

$$H - m(r)^{1/2} mc^2 := H_0$$

$$= \frac{p^2}{m^{1/2}(r)\left(1+m(r)\right) m}\left(1 + \frac{U}{m^{1/2}(r)\left(1+m(r)\right) mc^2}\right) + U \quad -(38)$$

) In the limit:

$$m(r) \to 1 \quad -(39)$$

and

$$\frac{U}{mc^2} \to 0 \quad -(40)$$

eq.(38) reduces to the classical:

$$H_0 = \frac{p^2}{2m} + U. \quad -(41)$$

In the limit:

$$m(r) \to 1 \quad -(42)$$

it reduces to the Dirac theory:

$$H_0 = H - mc^2 = \frac{p^2}{2m}\left(1 + \frac{U}{2mc^2}\right) + U \quad -(43)$$

Now denote:

$$f(r) := m^{1/2}(r)\left(1 + m(r)\right) \quad -(44)$$

and introduce the SU(2) basis:

$$H_0 = H - m(r)^{1/2} mc^2 \quad -(45)$$

$$= \frac{1}{2m}\underline{\sigma}\cdot\underline{p}\left(\frac{1}{f(r)}\left(1 + \frac{U}{f(r)mc^2}\right)\right)\underline{\sigma}\cdot\underline{p} + U$$

For the interaction between an electron and a proton:

$$U(r) = -\frac{m(r)^{1/2}e^2}{4\pi\epsilon_0 r} \quad -(46)$$

So:

7)

$$H_0 = \frac{1}{m} \underline{\sigma} \cdot \underline{p} \left( \frac{1}{f(r)} \left( 1 - \frac{e^2}{(1+m(r))\, 4\pi \epsilon_0 m c^2 r} \right) \right) \underline{\sigma} \cdot \underline{p} + U \quad -(47)$$

The theory quantizes to:

$$H_0 \phi = -\frac{i\hbar}{m} \underline{\sigma} \cdot \underline{\nabla} \left( \frac{1}{f(r)} \left( 1 - \frac{e^2}{4\pi \epsilon_0 m c^2 (1+m(r)) r} \right) \right) \underline{\sigma} \cdot \underline{p} \phi + U \phi$$

The hamiltonian (47) can be written as:

$$H_0 = H_1 + H_2 + U \quad -(48)$$

where

$$H_1 = \frac{1}{m} \underline{\sigma} \cdot \underline{p} \, \frac{1}{f(r)} \, \underline{\sigma} \cdot \underline{p} \quad -(49)$$

and

$$H_2 = -\frac{1}{m} \underline{\sigma} \cdot \underline{p} \, \frac{e^2}{4\pi \epsilon_0 m c^2 f(r) (1+m(r)) r} \, \underline{\sigma} \cdot \underline{p} + U \quad -(50)$$

i.e.

$$H_2 = -\frac{e^2}{4\pi \epsilon_0 m c^2} \underline{\sigma} \cdot \underline{p} \left( \frac{1}{f(r)(1+m(r)) r} \right) \underline{\sigma} \cdot \underline{p} + U \quad -(51)$$

1) The hamiltonian $H_1$ is the Schroedinger hamiltonian modified by m theory.

2) The hamiltonian $H_2$ is the fine structure hamiltonian modified by m theory.

8) Conclusions

The m theory modifies the Schroedinger H atom and also the Dirac H atom. In the Dirac theory:

$$f(r) = 2 \quad -(52)$$

and

$$n(r) = 1. \quad -(53)$$

So:

$$H_1 \psi = -\frac{\hbar^2 \nabla^2}{2m} \psi \quad -(54)$$

because

$$\underline{\sigma} \cdot \underline{p} \, \underline{\sigma} \cdot \underline{p} = p^2 \quad -(55)$$

but in the m theory:

$$H_1 \psi = -\frac{\hbar^2}{m} \underline{\sigma} \cdot \underline{\nabla} \left( \frac{1}{f(r)} \underline{\sigma} \cdot \underline{\nabla} \psi \right) \quad -(56)$$

and in general $\underline{\nabla}$ acts on $f_1(r)$.

So all the energy levels of the H atom are changed in m space. Similarly all the fine structure of the H atom is changed in m theory. In the Dirac theory the energy levels from eq. (54) are:

$$E_1 = -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d\tau, \quad -(57)$$

but in the m theory:

$$E_1 = -\frac{\hbar^2}{m} \int \psi^* \underline{\sigma} \cdot \underline{\nabla} \left( \frac{1}{f(r)} \underline{\sigma} \cdot \underline{\nabla} \psi \right) d\tau$$