(2): Demonstration of Self Consistency of the Theory

This is first demonstrated in flat or Minkowski spacetime and then in the rest frame in m space.

In flat spacetime the lagrangian is:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{\underline{r}}\cdot\dot{\underline{r}}}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(1)$$

and in plane polar coordinates (r, ϕ):

$$\dot{\underline{r}}\cdot\dot{\underline{r}} = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(2)$$

so eq. (1) is equivalent to:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(3)$$

The Euler Lagrange equation with Lagrange variable $\underline{r}$ is:

$$\underline{F} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}} = \frac{\partial\mathcal{L}}{\partial\underline{r}} = \underline{\nabla}\mathcal{L} \quad -(4)$$

So the force $\underline{F}$ is the gradient of the lagrangian.

In general:

$$\underline{\nabla}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\mathcal{L}}{\partial\phi}\underline{e}_\phi \quad -(5)$$

where $\underline{e}_r$ and $\underline{e}_\phi$ are the unit vectors of the plane polar system (r, ϕ). From eq. (1) the relativistic linear momentum is:

$$\underline{p} = \frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}} = \gamma m\dot{\underline{r}} \quad -(6)$$

Therefore the Euler Lagrange equation (4)

) given:

$$\frac{d}{dt}(\gamma m \underline{\dot{r}}) = \frac{\partial \mathcal{L}}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial \mathcal{L}}{\partial \phi}\underline{e}_\phi \quad -(7)$$

In plane polar coordinates:

$$\underline{\dot{r}} = \dot{r}\underline{e}_r + r\dot{\phi}\underline{e}_\phi \quad -(8)$$

$$\underline{\ddot{r}} = (\ddot{r} - r\dot{\phi}^2)\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\underline{e}_\phi \quad -(9)$$

The lagrangian has no dependence on ϕ so eq. (7) becomes: $-(10)$

$$m\dot{r}\frac{d\gamma}{dt} + m\gamma\underline{\ddot{r}} = \frac{\partial \mathcal{L}}{\partial r}\underline{e}_r = -\frac{mMG}{r^2}\underline{e}_r$$

i.e.

$$\left(m\dot{r}\frac{d\gamma}{dt} + m\gamma(\ddot{r} - r\dot{\phi}^2)\right)\underline{e}_r + \left(m\frac{d\gamma}{dt}r\dot{\phi} + m\gamma(r\ddot{\phi} + 2\dot{r}\dot{\phi})\right)\underline{e}_\phi = -\frac{mMG}{r^2}\underline{e}_r \quad -(11)$$

This equation means that:

$$\left(m\dot{r}\frac{d\gamma}{dt} + m\gamma(\ddot{r} - r\dot{\phi}^2)\right) = -\frac{mMG}{r^2} \quad -(12)$$

and

$$m\frac{d\gamma}{dt}r\dot{\phi} + m\gamma(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = 0 \quad -(13)$$

Eqs. (12) and (13) can be rewritten as:

$$\frac{d}{dt}(\gamma m \underline{\dot{r}}) = -\frac{mMG}{r^2}\underline{e}_r \quad -(14)$$

and

$$\frac{d}{dt}(\gamma m r^2 \dot{\phi}) = 0 \quad -(15)$$

Now consider the Lagrangian (8) wrt Lagrange variables r and ϕ, giving two Euler Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(16)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad -(17)$$

Eq. (16) gives:

$$\frac{d}{dt}\left(\gamma m \dot{r}\right) = -\frac{mMG}{r^2} + m\dot{\phi}^2\gamma$$

and Eq. (17) gives:

$$\frac{d}{dt}\left(\gamma m r^2 \dot{\phi}\right) = 0 \quad -(19)$$

which are the same as eqs. (14) and (15) Q.E.D.

The angular momentum in this spacetime is:

$$L = \gamma m r^2 \dot{\phi} \quad -(20)$$

and is a conserved constant of motion:

$$\frac{dL}{dt} = 0 \quad -(21)$$

Defining

$$\underline{p} = \gamma m \underline{\dot{r}}, \quad -(22)$$

the equations of motion are:

$$\underline{F} = \frac{d\underline{p}}{dt} = -\frac{mMG}{r^2}\underline{e}_r \quad -(23)$$

and

$$\frac{dL}{dt} = 0 \quad -(24)$$

It has been shown in previous work that eqs. (18) and (19) give a precessing orbit.

The hamiltonian of flat spacetime is:

$$H = \gamma m c^2 - \frac{m M G}{r} \quad -(25)$$

and is a conserved constant of motion:

$$\frac{dH}{dt} = 0 \quad -(26)$$

It follows that:

$$\frac{d}{dt}\left(\gamma m c^2\right) = \frac{d}{dt}\left(\frac{m M G}{r}\right) \quad -(27)$$

i.e.

$$m c^2 \frac{d\gamma}{dt} = \frac{d}{dr}\left(\frac{m M G}{r}\right)\frac{dr}{dt} = -\frac{m M G}{r^2}\dot{r} \quad -(28)$$

Now note:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} = \ddot{r}\frac{d\gamma}{dv} \quad -(29)$$

and

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} = \gamma^3 \frac{\dot{r}}{c^2} \quad -(30)$$

It follows that eq. (28) gives:

$$\underline{F} = m\gamma^3 \underline{\ddot{r}} = -\frac{m M G}{r^2}\underline{e}_r \quad -(31)$$

which is the Newtonian force in flat spacetime, Q.E.D.

It can be shown as follows that:

$$\frac{d}{dt}\left(\gamma \dot{r}\right) = \gamma^3 \ddot{r} \quad -(32)$$

because:

$$\frac{d}{dt}\left(\gamma\dot{r}\right) = \frac{d\gamma}{dt}\dot{r} + \gamma\ddot{r}$$

$$= \frac{d\gamma}{dv}\ddot{r}\dot{r} + \gamma\ddot{r}$$

$$= \ddot{r}\left(\frac{d\gamma}{dv}\dot{r} + \gamma\right)$$

$$= \ddot{r}\left(\gamma^3\frac{\dot{r}^2}{c^2} + \gamma\right) \quad -(33)$$

$$= \gamma^3\ddot{r}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right)$$

$$= \gamma^3\ddot{r}\left(\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right)$$

$$= \gamma^3\ddot{r}$$

Q.E.D.

It follows that eq. (26) leads to eq. (18) in a rigorously self consistent way:

$$\underline{F} = m\gamma^3\underline{\ddot{r}} = \frac{d}{dt}(\gamma m\underline{\dot{r}}) = -\frac{mMG}{r^2}\underline{e}_r \quad -(34)$$

It has been shown in previous work that eq. (34), solved simultaneously with eq. (19), leads to a precessing orbit.

In the classical limit:

$$\gamma = 1 \quad -(35)$$

so

$$\underline{F} = m\underline{\ddot{r}} = -\frac{mMG}{r^2}\underline{e}_r \quad -(36)$$

and

$$\frac{dL}{dt} = \frac{d}{dt}(mr^2\dot{\phi}) = 0 \quad -(37)$$

The numerical method has been checked to show that eqs. (36) and (37) produce conic section orbits, Q.E.D.