99(2): Expression for the Vacuum Electric Field in Terms of the Material Electric Field.

In electrostatics the observed potential is:

$$\phi(\underline{r}+\delta\underline{r}) = \phi(\underline{r}) + \phi(vac) \quad -(1)$$

so

$$\phi(vac) = \phi(\underline{r}+\delta\underline{r}) - \phi(\underline{r}) \quad -(2)$$

$$= \Delta\phi$$

Using the tensor Taylor series and isotropic averaging:

$$\langle\Delta\phi\rangle = \langle\Delta\phi\rangle^{(2)} + \langle\Delta\phi\rangle^{(4)} + \langle\Delta\phi\rangle^{(6)} \quad -(3)$$

Here:

$$\langle\Delta\phi\rangle^{(2)} = \frac{1}{6}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle \nabla^2\phi(\underline{r}) \quad -(4)$$

$$\langle\Delta\phi\rangle^{(4)} = \frac{1}{216}\langle(\delta\underline{r}\cdot\delta\underline{r})^2\rangle\left(\frac{\partial^4\phi(\underline{r})}{\partial x^4} + \frac{\partial^4\phi(\underline{r})}{\partial y^4} + \frac{\partial^4\phi(\underline{r})}{\partial z^4} + 6\left(\frac{\partial^4\phi(\underline{r})}{\partial y^2\partial z^2} + \frac{\partial^4\phi(\underline{r})}{\partial x^2\partial z^2} + \frac{\partial^4\phi(\underline{r})}{\partial x^2\partial y^2}\right)\right) \quad -(5)$$

$$\langle\Delta\phi\rangle^{(6)} = \frac{\langle(\delta\underline{r}\cdot\delta\underline{r})^3\rangle}{19440}\left(\frac{\partial^6\phi(\underline{r})}{\partial x^6} + \frac{\partial^6\phi(\underline{r})}{\partial y^6} + \frac{\partial^6\phi(\underline{r})}{\partial z^6} + 15\left(\frac{\partial^6\phi(\underline{r})}{\partial y^4\partial z^2} + \frac{\partial^6\phi(\underline{r})}{\partial y^2\partial z^4} + \frac{\partial^6\phi(\underline{r})}{\partial x^4\partial z^2} + \frac{\partial^6\phi(\underline{r})}{\partial x^4\partial y^2} + \frac{\partial^6\phi(\underline{r})}{\partial x^2\partial z^4} + \frac{\partial^6\phi(\underline{r})}{\partial x^2\partial y^4}\right) + 90\frac{\partial^6\phi(\underline{r})}{\partial x^2\partial y^2\partial z^2}\right) \quad -(6)$$

Similarly:

$$\underline{E}(vac) = \underline{E}(\underline{r}+\delta\underline{r}) - \underline{E}(\underline{r}) \quad -(7)$$

$$= \Delta\underline{E}$$

So

$$\underline{E}(\text{vac}) = \langle \Delta \underline{E} \rangle^{(2)} + \langle \Delta \underline{E} \rangle^{(4)} + \langle \Delta \underline{E} \rangle^{(6)} + \ldots \quad -(8)$$

For simplicity of argument consider the sum to second order:

$$\phi(\text{vac}) = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 \phi(\underline{r}) \quad -(9)$$

$$\underline{E}(\text{vac}) = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \left(\nabla^2 E_x(\underline{r}) \underline{i} + \nabla^2 E_y(\underline{r}) \underline{j} + \nabla^2 E_z(\underline{r}) \underline{k} \right) \quad -(10)$$

So:

$$\underline{E}(\text{vac}) = \frac{\phi(\text{vac})}{\nabla^2 \phi(\underline{r})} \left(\nabla^2 E_{0x} \underline{i} + \nabla^2 E_{0y} \underline{j} + \nabla^2 E_{0z} \underline{k} \right) \quad -(11)$$

i.e

$$E_x(\text{vac}) = \frac{\phi(\text{vac})}{\nabla^2 \phi(\underline{r})} \nabla^2 E_x(\underline{r}) \quad -(12)$$

i.e.

$$\frac{E_x(\text{vac})}{\phi(\text{vac})} = \frac{\nabla^2 E_x(\underline{r})}{\nabla^2 \phi(\underline{r})} \quad -(13)$$

$$\frac{E_y(\text{vac})}{\phi(\text{vac})} = \frac{\nabla^2 E_y(\underline{r})}{\nabla^2 \phi(\underline{r})} \quad -(14)$$

$$\frac{E_z(\text{vac})}{\phi(\text{vac})} = \frac{\nabla^2 E_z(\underline{r})}{\nabla^2 \phi(\underline{r})} \quad -(15)$$

The ratios (13) to (15) do not require knowledge of $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$

In ECE theory:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi \quad -(16)$$

which can be interpreted as:

$$\underline{E}(\underline{r}+\delta\underline{r}) = \underline{E}(\underline{r}) + \underline{E}(\text{vac}) \quad -(17)$$

in which:

$$\underline{E}(\underline{r}) = -\underline{\nabla}\phi(\underline{r}) \quad -(18)$$

and

$$\underline{E}(\text{vac}) = \underline{\omega}\,\phi(\text{vac}) \quad -(19)$$

It follows from eq. (19) that:

$$\omega_X = \frac{E_X(\text{vac})}{\phi_X(\text{vac})} = \frac{\nabla^2 E_X(\underline{r})}{\nabla^2\phi(\underline{r})} \quad -(20)$$

$$\omega_Y = \frac{E_Y(\text{vac})}{\phi_Y(\text{vac})} = \frac{\nabla^2 E_Y(\underline{r})}{\nabla^2\phi(\underline{r})} \quad -(21)$$

$$\omega_Z = \frac{E_Z(\text{vac})}{\phi_Z(\text{vac})} = \frac{\nabla^2 E_Z(\underline{r})}{\nabla^2\phi(\underline{r})} \quad -(22)$$

in which:

$$\underline{E}(\underline{r}) = -\underline{\nabla}\phi(\underline{r}) \quad -(23)$$

Now use the Poisson equation:

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \quad -(24)$$

where ρ is the material charge density. Eq. (24)

) is interpreted as:

$$\nabla^2 \phi(\underline{r}) = -\rho / \epsilon_0 \quad -(25)$$

where $\phi(\underline{r})$ is the potential in the absence of the vacuum. The highly developed subject that analyses solutions of the Poisson equation means that $\phi(\underline{r})$ can be found for a given charge density ρ. Finally:

$$\underline{E}(\underline{r}) = -\underline{\nabla}\phi(\underline{r}) \quad -(26)$$

can be found, and the three spin connection components (20) to (22) can be found.