399(3): Complete Expression for the Spin Connection

Consider the definition of the spin connection:

$$\langle \underline{E}(vac) \rangle = \underline{\omega} \langle \phi(vac) \rangle \quad -(1)$$

where:

$$\langle \underline{E}(vac) \rangle = \langle \underline{E}(vac) \rangle^{(2)} + \langle \underline{E}(vac) \rangle^{(4)} + \langle \underline{E}(vac) \rangle^{(6)} + \cdots \quad -(2)$$

and

$$\langle \phi(vac) \rangle = \langle \phi(vac) \rangle^{(2)} + \langle \phi(vac) \rangle^{(4)} + \langle \phi(vac) \rangle^{(6)} + \cdots \quad -(3)$$

In eqs. (1) to (3):

$$\langle \phi(vac) \rangle^{(2)} = \langle \Delta\phi \rangle^{(2)} = \langle \delta\underline{r} \cdot \delta\underline{r} \rangle f^{(2)}(\phi) \quad -(4)$$

$$\langle \phi(vac) \rangle^{(4)} = \langle \Delta\phi \rangle^{(4)} = \langle (\delta\underline{r} \cdot \delta\underline{r})^2 \rangle f^{(4)}(\phi) \quad -(5)$$

$$\langle \phi(vac) \rangle^{(6)} = \langle \Delta\phi \rangle^{(6)} = \langle (\delta\underline{r} \cdot \delta\underline{r})^3 \rangle f^{(6)}(\phi) \quad -(6)$$

$$\langle E_x(vac) \rangle^{(2)} = \langle \Delta E_x \rangle^{(2)} = \langle \delta\underline{r} \cdot \delta\underline{r} \rangle f^{(2)}(E_x) \quad -(7)$$

$$\langle E_x(vac) \rangle^{(4)} = \langle \Delta E_x \rangle^{(4)} = \langle (\delta\underline{r} \cdot \delta\underline{r})^2 \rangle f^{(4)}(E_x) \quad -(8)$$

$$\langle E_x(vac) \rangle^{(6)} = \langle \Delta E_x \rangle^{(6)} = \langle (\delta\underline{r} \cdot \delta\underline{r})^3 \rangle f^{(6)}(E_x) \quad -(9)$$

and so on. Here the functions $f^{(2)}, f^{(4)}, f^{(6)}, \ldots$ are defined by the Taylor series.

It follows that:

$$\frac{\langle \phi(vac) \rangle^{(2)}}{\langle E_x(vac) \rangle^{(2)}} = \frac{f^{(2)}(\phi)}{f^{(2)}(E_x)} \quad -(10)$$

$$\frac{\langle \phi(vac) \rangle^{(4)}}{\langle E_x(vac) \rangle^{(4)}} = \frac{f^{(4)}(\phi)}{f^{(4)}(E_x)} \quad -(11)$$

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$$\frac{\langle \phi(\mathrm{vac}) \rangle^{(6)}}{\langle E_x(\mathrm{vac}) \rangle^{(6)}} = \frac{f^{(6)}(\phi)}{f^{(6)}(E_x)} \qquad -(12)$$

$$\vdots$$

It follows that:

$$\langle \phi(\mathrm{vac}) \rangle^{(2)} = x_1 \langle E_x(\mathrm{vac}) \rangle^{(2)} \qquad -(13)$$

$$\langle \phi(\mathrm{vac}) \rangle^{(4)} = x_2 \langle E_x(\mathrm{vac}) \rangle^{(4)} \qquad -(14)$$

$$\langle \phi(\mathrm{vac}) \rangle^{(6)} = x_3 \langle E_x(\mathrm{vac}) \rangle^{(6)} \qquad -(15)$$

where

$$x_1 = \frac{f^{(2)}(\phi)}{f^{(2)}(E_x)}; \quad x_2 = \frac{f^{(4)}(\phi)}{f^{(4)}(E_x)}; \quad x_3 = \frac{f^{(6)}(\phi)}{f^{(6)}(E_x)} \qquad -(16)$$

It follows that:

$$\langle E_x(\mathrm{vac}) \rangle^{(2)} + \langle E_x(\mathrm{vac}) \rangle^{(4)} + \langle E_x(\mathrm{vac}) \rangle^{(6)} + \cdots$$

$$= \omega_x \left(x_1 \langle E_x(\mathrm{vac}) \rangle^{(2)} + x_2 \langle E_x(\mathrm{vac}) \rangle^{(4)} + x_3 \langle E_x(\mathrm{vac}) \rangle^{(6)} + \cdots \right)$$

$$:= x\,\omega_x \left(\langle E_x(\mathrm{vac}) \rangle^{(2)} + \langle E_x(\mathrm{vac}) \rangle^{(4)} + \langle E_x(\mathrm{vac}) \rangle^{(6)} + \cdots \right) \qquad -(17)$$

So:

$$x = \frac{x_1 \langle E_x(\mathrm{vac}) \rangle^{(2)} + x_2 \langle E_x(\mathrm{vac}) \rangle^{(4)} + \cdots}{\langle E_x(\mathrm{vac}) \rangle^{(2)} + \langle E_x(\mathrm{vac}) \rangle^{(4)} + \cdots} \qquad -(18)$$

and

$$\boxed{\omega_x = \frac{1}{x}} \qquad -(19)$$