396(5): Calculating the Root Mean Square Vacuum Fluctuation Used in Lamb Shift Theory

The use of mode theory for the vacuum gives:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{1}{2\epsilon_0 \pi^2} \frac{e^2}{\hbar c} \left(\frac{\hbar}{mc} \right)^2 \log_e \frac{4\epsilon_0 \hbar c}{e^2} \quad - (1)$$

in which the Compton wavelength of the electron is:

$$\lambda = \frac{h}{mc} = 2.426309 \times 10^{-12} \text{ m} \quad - (2)$$

so

$$\frac{\hbar}{mc} = \frac{\lambda}{2\pi} = 3.8616 \times 10^{-13} \text{ m} \quad - (3)$$

The fine structure constant is:

$$\alpha = \frac{e^2}{4\pi \hbar c \epsilon_0} = 0.007297351 \quad - (4)$$

so

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2\alpha}{\pi} \left(\frac{\hbar}{mc} \right)^2 \log_e \frac{4\epsilon_0 \hbar c}{e^2} \quad - (5)$$

$$= \frac{2 \times 0.007297351}{\pi} \times 3.8616^2 \times 10^{-26} \log_e \frac{4\epsilon_0 \hbar c}{e^2}$$

$$= 6.9276 \times 10^{-28} \log_e \frac{4\epsilon_0 \hbar c}{e^2}$$

in which

$$\frac{4\epsilon_0 \hbar c}{e^2} = \frac{1}{\pi \alpha} = 43.62 \quad - (6)$$

so

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2\alpha}{\pi} \left(\frac{\hbar}{mc} \right)^2 \log_e 43.62$$

$$= 6.9276 \times 3.776 \times 10^{-28} \text{ m}^2 \quad - (7)$$

$$= 26.16 \times 10^{-28} \text{ m}^2$$

so

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle^{1/2} = 5.115 \times 10^{-14} \text{ metres} \quad - (8)$$

The expectation values of orbital radii in atomic H are as follows:

$$\langle r \rangle (1s) = \frac{3}{2} a_0 \quad -(9)$$

$$\langle r \rangle (2s) = 6 a_0 \quad -(10)$$

$$\langle r \rangle (3s) = \frac{27}{2} a_0 \quad -(11)$$

where the Bohr radius is:

$$a_0 = 5.29177 \times 10^{-11} m \quad -(12)$$

So

$$\langle r \rangle (2s) = 3.1751 \times 10^{-10} m \quad -(13)$$

and

$$\boxed{\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle^{1/2}}{\langle r \rangle (2s)} = \frac{5.115}{3.1751} \times 10^{-4} = 1.611 \times 10^{-4}} \quad -(14)$$

The change in Coulombic potential energy U due to the Lamb shift is:

$$\langle \Delta U \rangle = \frac{\alpha^5 mc^2}{6\pi} \log_e \frac{1}{\pi \alpha} \quad -(15)$$

where m is the mass of the electron:

$$m = 9.10953 \times 10^{-31} kg \quad -(16)$$

To second order in the tensor Taylor series:

$$\langle \Delta U \rangle = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 U \quad -(17)$$

It follows that:

$$\langle \Delta U \rangle = \frac{1}{6} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \int \phi^* \nabla^2 U \phi \, d\tau \quad -(18)$$

In evaluating this expectation value, the Dirac delta function is used:

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi \, \delta_D(r) \quad -(19)$$

So

$$\left\langle \nabla^2 \left(-\frac{e^2}{4\pi \epsilon_0 r} \right) \right\rangle = -\frac{e^2}{4\pi \epsilon_0} \int \phi^*(r) \nabla^2 \left(\frac{1}{r} \right) \phi(r) \, d\tau$$

$$= \frac{e^2}{\epsilon_0} |\phi(0)|^2 \quad -(20)$$

where $\phi(0)$ is the wave function at the nucleus.

Therefore the Lamb shift is given by:

$$\boxed{\langle \Delta U \rangle = \frac{1}{6} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \frac{e^2}{\epsilon_0} |\phi(0)|^2 \quad -(21)}$$

For the 2S wave function of atomic H:

$$\phi_{2S}(0) = \frac{1}{(8\pi a_0^3)^{1/2}} \quad -(22)$$

So

$$\langle \Delta U \rangle_{2S} = \frac{\alpha^5 mc^2}{6\pi} \log_e \left(\frac{1}{\pi \alpha} \right) \quad -(23)$$

For the 2P wave function of atomic H:

$$\phi_{2P}(0) = 0 \quad -(24)$$

and

$$\langle \Delta U \rangle_{2P} = 0 \quad -(25)$$

Therefore the degeneracy of the ϕ_{2S} and ϕ_{2P} states

4) is lifted. In the Dirac theory of H, their energies are the same.

From eq. (23):

$$\langle \Delta U \rangle_{2S} = 0.007297351^5 \times 9.10953 \times 10^{-31} \times 2.997925^2 \times 10^{16} \times \frac{3.776}{6\pi} \text{ Joules}$$

$$= 7.297351^5 \times 10^{-15} \times 9.10953 \times 10^{-31} \times 2.997925^2 \times 10^{16} \times \frac{3.776}{6\pi}$$

$$= 7.297351^5 \times 9.10953 \times 2.997925^2 \times \frac{3.776}{6\pi} \times 10^{-30}$$

$$= \frac{20{,}693.13 \times 9.10953 \times 8.9876 \times 3.776}{6\pi} \times 10^{-30}$$

$$= 3.3939 \times 10^{-25} \text{ Joules} \quad -(26)$$

So

$$\boxed{\begin{aligned} \langle \Delta U \rangle_{2S} &= 3.3939 \times 10^{-25} \text{ Joules} \quad &-(27) \\ \langle \Delta U \rangle_{2P} &= 0 \quad &-(28) \end{aligned}}$$

and this is due to:

$$\boxed{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = 2.616 \times 10^{-27} \, m^2} \quad -(29)$$

The Coulombic energy between the proton and the 2S electron can be estimated from:

$$U = -\frac{e^2}{4\pi\epsilon_0 \langle r \rangle^2} \quad -(30)$$

where

$$\langle r \rangle = 3.1751 \times 10^{-10} \, m \quad -(31)$$

Therefore the Coulombic energy is:

5)
$$U = -\frac{1.60219^2 \times 10^{-38}}{4\pi \times 8.85419 \times 10^{-12} \times 3.1751^2 \times 10^{-20}}$$

$$= -2.289 \times 10^{-3} \times 10^{-6}$$

$$= -2.289 \times 10^{-9} \text{ Joules.} \quad -(32)$$

Therefore

$$\boxed{\frac{\langle \Delta U \rangle_{2S}}{|U|_C} = \frac{3.3939 \times 10^{-25}}{2.289 \times 10^{-9}} = 1.483 \times 10^{-16}} \quad -(33)$$

Therefore the change in Coulombic energy is very small. The relative change in proton to electron radius is much larger, and is given by eq. (14).

The energy levels of the H atom are given by

$$E = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} \quad -(34)$$

where n is the principal quantum number. For the 2S level:

$$n = 2 \quad -(35)$$

The energy levels (34) can be expressed as:

$$E = -\frac{mc^2}{2}\frac{\alpha^2}{n^2} \quad -(36)$$

So for $n = 2$:

$$E = -\frac{1}{8}mc^2\alpha^2 \quad -(37)$$

$$= -5.45 \times 10^{-19} \text{ Joules}$$

6) Therefore:

$$\frac{\langle \Delta U \rangle_{2S}}{|E|} = \frac{3.3939 \times 10^{-25}}{5.45 \times 10^{-19}} \quad -(38)$$

$$= 6.23 \times 10^{-7}$$

This is the ratio of the Lamb shift to the 2S energy level of H, and it is observed experimentally with precision. It is due to:

$$\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle^{1/2}}{\langle r \rangle (2S)} = 1.611 \times 10^{-4} \quad -(39)$$

The methodology of the Lamb shift is to start with the classical Coulomb potential and to calculate a correction of the Dirac equation. The wave functions used are the non-relativistic wave functions of H. The correction is:

$$\Delta U = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 U \quad -(40)$$

where

$$U = -\frac{e^2}{4\pi \epsilon_0 r^2} \quad -(41)$$

is the Coulombic energy of attraction between the proton and electron in the H atom. The expectation value is worked out with the Schroedinger wave functions ϕ:

$$\langle \nabla^2 U \rangle = \int \phi^* \nabla^2 U \phi \, d\tau \quad -(42)$$

So the theory starts with the Schroedinger H atom and its wave functions ϕ, and corrects it

7) with the Tensor Taylor series (40), using a model theory for $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$.

Therefore the Schroedinger theory of the H atom is a theory in the hypothetical absence of the vacuum. More accurately, the Dirac H atom should be used as the starting point, with the Dirac wavefunctions. The Dirac theory is also a theory in the hypothetical absence of the vacuum.

To extend this famous methodology to the rest of physics it is necessary to chose a theory in the absence of the vacuum, a theory that gives known results, and to correct it with a tensor Taylor series to give results that can be measured experimentally.

For example, in Note 396(3), the theory in the absence of the vacuum is the Newtonian universal gravitation, with force law:

$$\underline{F} = -mMG\frac{\underline{r}}{r^3} \quad -(42)$$

This is corrected for the influence of the vacuum with the tensor Taylor expansion to give:

$$\underline{F}(\underline{r} + \delta\underline{r}) = -mMG\frac{\underline{r}}{r^3} + \Delta\underline{F} \quad -(43)$$

and it is known experimentally that this equation must give precession due to the vacuum. The latter to fourth and sixth order in the Taylor series produces a shoulder in the inverse square law (42).