396(2): Coulomb Potential and Tensor Taylor Series

In this case:

$$f = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(1)$$

in the usual notation. In the theory of the Lamb shift f is the Coulomb potential between an electron and proton in an H atom. In the Lamb shift theory:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta_D(r) \quad -(2)$$

where $\delta_D(r)$ is the Dirac delta function. Then to second order:

$$\langle \Delta\phi^{(2)} \rangle = \frac{1}{6}\langle \delta\underline{r}\cdot\delta\underline{r}\rangle \left\langle \nabla^2\left(-\frac{e^2}{4\pi\epsilon_0 r}\right)\right\rangle \quad -(3)$$

where the expectation value is:

$$\left\langle \nabla^2\left(-\frac{e^2}{4\pi\epsilon_0 r}\right)\right\rangle = -\frac{e^2}{4\pi\epsilon_0}\int \psi^*(r)\,\nabla^2\left(\frac{1}{r}\right)\psi(r)\,dr$$

$$= \frac{e^2}{\epsilon_0}\,|\psi(0)|^2 \quad -(4)$$

This is a quantum mechanical calculation.

On the classical level, isotropic averaging of the second term in the tensor Taylor series gives the following result, as in Note 396(1):

$$\Delta f^{(2)} = \frac{1}{2!}\left\langle \left(\delta X\frac{\partial}{\partial X} + \delta Y\frac{\partial}{\partial Y} + \delta Z\frac{\partial}{\partial Z}\right)\left(\delta X\frac{\partial f}{\partial X} + \delta Y\frac{\partial f}{\partial Y} + \delta Z\frac{\partial f}{\partial Z}\right)\right\rangle$$

$$= \frac{1}{6}\langle \delta\underline{r}\cdot\delta\underline{r}\rangle\,\nabla^2 f \quad -(5)$$

in which:

2)

$$\langle \delta x^2 \rangle = \langle \delta y^2 \rangle = \langle \delta z^2 \rangle = \frac{1}{3} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(6)$$

To fourth order:

$$\Delta f^{(4)} = \frac{1}{4!} \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right) \left[\left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right) \right.$$

$$\left. \left[\left(\partial x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right) \left(\partial x \frac{\partial f}{\partial x} + \partial y \frac{\partial f}{\partial y} + \delta z \frac{\partial f}{\partial z} \right) \right] \right] \quad -(7)$$

By ordinary vector analysis:

$$\nabla^2 f = 0 \quad -(8)$$

so

$$\Delta f^{(2)} = 0 \quad -(9)$$

However, computer algebra shows that:

$$\Delta f^{(4)} \neq 0 \quad -(10)$$

even if the result (8) is assumed.

When Dirac first introduced the delta function it was rejected by contemporary mathematicians because of the contradiction between eqs. (2) and the result of vector algebra:

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \quad -(11)$$

However, later developments in mathematics gave a rigorous basis for the Dirac delta function.

In order to proceed, it is possible to select expressions such as:

$$\langle \Delta\phi \rangle = \frac{1}{4!} \langle \Delta\phi^{(4)} \rangle + \frac{1}{6!} \langle \Delta\phi^{(6)} \rangle + \frac{1}{8!} \langle \Delta\phi^{(8)} \rangle + \dots \quad -(12)$$

The fourth order correction $\langle \Delta\phi^{(4)} \rangle$ is found by using

$$f = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(13)$$

in eq. (7). This has been done already by coauthor Horst Eckardt. Similarly, the sixth and eighth order corrections can be found from computer algebra.

Note carefully that Eq. (12) does not use the Dirac delta function.

In order to reproduce the Lamb shift, the expectation value of $\langle \Delta\phi \rangle$ from eq. (12) must be the same as the expectation value of $\langle \Delta\phi^{(2)} \rangle$ from eqs. (3) and (4):

$$\int \phi^* \langle \Delta\phi^{(2)} \rangle \phi \, d\tau = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \frac{e^2}{\epsilon_0} |\phi(0)|^2 \quad -(14)$$

$$= \int \phi^* \langle \Delta\phi \rangle \phi \, d\tau$$

Using mode theory:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{1}{2\epsilon_0 \pi^2} \frac{e^2}{\hbar c} \left(\frac{\hbar}{mc} \right)^2 \log_e \left(\frac{4\epsilon_0 \hbar c}{e^2} \right) . \quad -(15)$$

For the 2S wavefunction of atomic H:

$$\phi_{2S}(0) = \frac{1}{(8\pi a_0^3)^{1/2}} \quad -(16)$$

4) The experimental result for the Lamb shift is:

$$\langle \Delta \phi \rangle = \frac{\alpha^5 mc^2}{6\pi} \log_e \left(\frac{1}{\pi \alpha} \right) \quad -(17)$$

where α is the fine structure constant. So, using the Taylor series:

$$\langle \Delta \phi \rangle = \frac{1}{2!} \langle \Delta \phi^{(2)} \rangle + \frac{1}{4!} \langle \Delta \phi^{(4)} \rangle + \frac{1}{6!} \langle \Delta \phi^{(6)} \rangle + \frac{1}{8!} \langle \Delta \phi^{(8)} \rangle + \dots \quad -(18)$$

$$= \frac{\alpha^5 mc^2}{6\pi} \log_e \left(\frac{1}{\pi \alpha} \right)$$

This series can be worked out without the use of the Dirac delta function. It is known from computer algebra that

$$\langle \Delta \phi^{(2)} \rangle = 0 \quad -(19)$$

but

$$\langle \Delta \phi^{(4)} \rangle \neq 0 \quad -(20)$$

and the experimental result for the vacuum fluctuations found from eq. (18).