395(3): Effects of the Vacuum in Electron - Electron Spin - Spin Interaction

The interaction energy is:

$$E = - \underline{m}_1 \cdot \underline{B}_2 \quad -(1)$$

where $\underline{m}_1$ is the magnetic dipole moment of electron 1, and where $\underline{B}_2$ is the magnetic flux density of electron 2. As in UFT394, the magnetic flux density is isotropically averaged in various approximations.

The Dipole Fields

In the second order approximation this is:

$$\langle \underline{B}_D \rangle^{(2)} = \frac{\mu_0}{4\pi r^3} \left(\frac{3 \underline{r} (\underline{m} \cdot \underline{r})}{r^2} - \underline{m} \right) - \frac{\mu_0}{4\pi r^5} \langle \delta\underline{r} \cdot \delta\underline{r} \rangle \left(\frac{35}{2} \frac{\underline{r} (\underline{m} \cdot \underline{r})}{r^2} - \frac{5}{2} \underline{m} \right) - \frac{5\mu_0}{8\pi r^7} \langle (\delta\underline{r} \cdot \delta\underline{r})^2 \rangle \underline{m} \,. \quad -(2)$$

It is convenient to rearrange this as:

$$\langle \underline{B}_D \rangle^{(2)} = \frac{\mu_0}{4\pi r^3} \left[\frac{\underline{r} (\underline{m}_2 \cdot \underline{r})}{r^2} \left(1 - \frac{35}{2} \frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{r^2} \right) - \underline{m}_2 \left(1 - \frac{5}{2} \left(\frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{r^2} + \frac{\langle (\delta\underline{r} \cdot \delta\underline{r})^2 \rangle}{r^4} \right) \right) \right] \quad -(3)$$

The interaction energy is therefore:

$$En = - \underline{m}_1 \cdot \langle \underline{B}_D \rangle^{(2)} = \frac{\mu_0}{4\pi r^3} \left[\underline{m}_1 \cdot \underline{m}_2 \left(1 - \frac{5}{2} \left(\frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{r^2} + \frac{\langle (\delta\underline{r} \cdot \delta\underline{r})^2 \rangle}{r^4} \right) \right) - 3 \underline{m}_1 \cdot \hat{\underline{r}} \hat{\underline{r}} \cdot \underline{m}_2 \left(1 - \frac{35}{2} \frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{r^2} \right) \right] \quad -(4)$$

2) The result in the absence of the vacuum is:

$$E_n = \frac{\mu_0}{4\pi r^3}\left(\underline{m}_1 \cdot \underline{m}_2 - 3\underline{m}_1 \cdot \hat{\underline{r}}\,\hat{\underline{r}} \cdot \underline{m}_2\right) \quad -(5)$$

where

$$\hat{\underline{r}} = \frac{\underline{r}}{|\underline{r}|} \quad -(6)$$

When spins are aligned in the Z axis, the quantized energy of interaction is:

$$E = \frac{\mu_0 \hbar^2 e^2}{4\pi m^2 r^3} m_{1s} m_{2s} \left[1 - \frac{5}{2}\left(\frac{\langle \delta\underline{r} \cdot \delta\underline{r}\rangle}{r^2} + \frac{\langle(\delta\underline{r} \cdot \delta\underline{r})^2\rangle}{r^4}\right) - 3\cos^2\theta\left(1 - \frac{35}{2}\frac{\langle \delta\underline{r} \cdot \delta\underline{r}\rangle}{r^2}\right)\right] \quad -(7)$$

in which:

$$m_{1s} = -\frac{1}{2}, \frac{1}{2} \quad -(8)$$

$$m_{2s} = -\frac{1}{2}, \frac{1}{2} \quad -(9)$$

and

$$\cos\theta = Z/r \quad -(10)$$

So the spin spin splitting is affected by the vacuum via the averages $\langle \delta\underline{r} \cdot \delta\underline{r}\rangle$ and $\langle(\delta\underline{r} \cdot \delta\underline{r})^2\rangle$

In the fourth order approximation:

$$\langle \underline{B}_D\rangle^{(4)} = \frac{\mu_0}{4\pi r^3}\left(\frac{3\underline{r}(\underline{m} \cdot \underline{r})}{r^2} + \frac{1435}{24}\frac{\langle(\delta\underline{r} \cdot \delta\underline{r})^2\rangle}{r^4}\frac{\underline{r}(\underline{m} \cdot \underline{r})}{r^2}\right)$$

$$-\frac{\mu_0}{4\pi r^3}\underline{m} + \frac{\mu_0}{4\pi r^3}\frac{\langle(\delta\underline{r} \cdot \delta\underline{r})^2\rangle}{r^4}\left(\frac{35}{24}\underline{m} - \frac{35}{r^2}\begin{bmatrix} X^2 & 0 & 0 \\ 0 & Y^2 & 0 \\ 0 & 0 & Z^2 \end{bmatrix}\underline{m}\right)$$

$$+\frac{\mu_0}{4\pi r^3}\frac{\langle(\delta\underline{r} \cdot \delta\underline{r})^3\rangle}{r^6} \cdot \frac{35}{8}\underline{m} \quad -(11)$$

$$= \frac{\mu_0}{4\pi r^3}\left[\frac{3\underline{r}(\underline{m}\cdot\underline{r})}{r^2}\left(1 + \frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^2\rangle}{r^4}\frac{1435}{72}\right)\right. \quad -(12)$$

$$\left. - \underline{m}\left(1 - 35\frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^2\rangle}{r^4}\left(\frac{1}{24} - \frac{1}{r^2}\begin{bmatrix} X^2 & 0 & 0 \\ 0 & Y^2 & 0 \\ 0 & 0 & Z^2 \end{bmatrix}\right) - \frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^3\rangle}{8r^6}\right)\right]$$

The classical interaction energy is:

$$E_n = -\underline{m}_1 \cdot \langle \underline{B}_D \rangle^{(4)}$$

$$= \frac{\mu_0}{4\pi r^3}\left[\underline{m}_1\cdot\underline{m}_2\left(1 - 35\frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^2\rangle}{r^4}\left(\frac{1}{24} - \frac{1}{r^2}\begin{bmatrix} X^2 & 0 & 0 \\ 0 & Y^2 & 0 \\ 0 & 0 & Z^2 \end{bmatrix}\right) + \frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^3\rangle}{8r^6}\right)\right. \quad -(13)$$

$$\left. - 3\underline{m}_1\cdot\underline{\hat{r}}\,\underline{\hat{r}}\cdot\underline{m}_2\left(1 + \frac{1435}{72}\frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^2\rangle}{r^4}\right)\right]$$

In spherical polar coordinates:

$$\frac{X}{r} = \sin\theta\cos\phi \,;\ \frac{Y}{r} = \sin\theta\sin\phi \,;\ \frac{Z}{r} = \cos\phi \quad -(14)$$

When spins are aligned in the Z axis, the quantized energy of interaction is: $\quad -(15)$

$$E = \frac{\mu_0 \hbar^2 e^2}{4\pi m^2 r^3} m_{1s} m_{2s}\left(1 - 35\frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^2\rangle}{r^4}\left(\frac{1}{24} - \frac{1}{r^2}\begin{bmatrix} X^2 & 0 & 0 \\ 0 & Y^2 & 0 \\ 0 & 0 & Z^2 \end{bmatrix}\right) + \frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^3\rangle}{8r^6} - 3\cos^2\theta\left(1 + \frac{1435}{72}\frac{\langle(\underline{\delta r}\cdot\underline{\delta r})^2\rangle}{r^4}\right)\right)$$

and the spin spin splitting is affected by the vacuum.

4) The Contact Fields

In a second and fourth order approximation, the contact fields are respectively:

$$\langle \underline{B}_c \rangle^{(2)} = \frac{\mu_0}{4\pi} \underline{m} \left(10 \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^5} + 15 \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle}{2 r^7} \right) \quad -(16)$$

and

$$\langle \underline{B}_c \rangle^{(4)} = -\frac{\mu_0}{4\pi} \underline{m} \left(\frac{105}{2 r^7} \langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle + 105 \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle}{8 r^9} \right) \quad -(17)$$

The classical interaction energy in each case is:

$$E_n^{(2)} = - \underline{m}_1 \cdot \langle \underline{B}_c \rangle^{(2)} \quad -(18)$$

and

$$E_n^{(4)} = - \underline{m}_1 \cdot \langle \underline{B}_c \rangle^{(4)} \quad -(19)$$

So:

$$E_n^{(2)} = -\frac{\mu_0}{4\pi r^3} \underline{m}_1 \cdot \underline{m}_2 \left(10 \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} + \frac{15}{2} \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle}{r^4} \right) \quad -(20)$$

and

$$E_n^{(4)} = \frac{\mu_0}{4\pi r^3} \underline{m}_1 \cdot \underline{m}_2 \left(\frac{105}{2 r^4} \langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle + 105 \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle}{8 r^6} \right) \quad -(21)$$

In the absence of the vacuum, these energies are zero unless the Dirac delta function is used.

For aligned spins, the quantized energy levels from contact interaction are given from:

$$\underline{m}_1 \cdot \underline{m}_2 \, \phi = \frac{e^2 \hbar^2}{m^2} m_{1s} m_{2s} \, \phi \quad -(22)$$

where

$$m_{1s} = \frac{1}{2}, -\frac{1}{2}, \quad m_{2s} = \frac{1}{2}, -\frac{1}{2} \quad -(23)$$

So the second order energy levels are:

$$E_n^{(2)} = -\frac{\mu_0 e^2 \hbar^2}{4\pi m^2 r^3} m_{1s} m_{2s} \left(10 \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} + \frac{15}{2} \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle}{r^4} \right) \quad -(24)$$

The total interaction energy is the sum of the dipole field and the contact field.