313(3): Maclaurin Series Expansion for the Effect of the Vacuum.

The Maclaurin series is named after Colin Maclaurin, who became a full professor in Aberdeen at the age of 19.

Consider:

$$f(\underline{r}+\delta\underline{r}) = f(\underline{r}) + (\underline{r}+\delta\underline{r})\cdot\underline{\nabla} f + \frac{1}{2}(\underline{r}+\delta\underline{r})^2\nabla^2 f + \frac{1}{6}(\underline{r}+\delta\underline{r})^3\nabla^3 f + \frac{1}{24}(\underline{r}+\delta\underline{r})^4\nabla^4 f + \dots \quad -(1)$$

where f is any scalar.

At

$$\underline{r} = \underline{0} \quad -(2)$$

$$\Delta f = f(\delta\underline{r}) - f(\underline{0}) = \delta\underline{r}\cdot\underline{\nabla} f + \frac{1}{2}\delta\underline{r}\cdot\delta\underline{r}\,\nabla^2 f + \frac{1}{6}(\delta\underline{r}\cdot\delta\underline{r})\,\delta\underline{r}\cdot\underline{\nabla}(\nabla^2 f) + \frac{1}{24}(\delta\underline{r}\cdot\delta\underline{r})(\delta\underline{r}\cdot\delta\underline{r})\nabla^4 f + \dots \quad -(3)$$

On isotropic averaging:

$$\langle\delta\underline{r}\rangle = \underline{0} \quad -(4)$$

So

$$\boxed{\langle\Delta f\rangle = \frac{1}{2!}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\nabla^2 f + \frac{1}{4!}\langle\delta\underline{r}\cdot\delta\underline{r}\;\delta\underline{r}\cdot\delta\underline{r}\rangle\nabla^4 f + \frac{1}{6!}\langle\delta\underline{r}\cdot\delta\underline{r}\;\delta\underline{r}\cdot\delta\underline{r}\;\delta\underline{r}\cdot\delta\underline{r}\rangle\nabla^6 f + \dots \quad -(5)}$$

This is a powerful and useful result, which can be

) used to calculate the average change $\langle \Delta f \rangle$ in any scalar function due to vacuum fluctuations.

To calculate the Lamb shift for example:

$$f = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(6)$$

and

$$\langle \delta\underline{r}\cdot\delta\underline{r}\rangle = \frac{1}{2\epsilon_0\pi^2}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\log_e\left(\frac{4\epsilon_0\hbar c}{e^2}\right) \quad -(7)$$

using mode theory similar to the one used in QED.

In order to calculate $\langle \delta\underline{r}\cdot\delta\underline{r}\rangle$, one is not restricted to mode theory. For example, the vacuum can be thought of as an ensemble of particles, and statistical mechanics and Brownian motion theory can be applied. The vacuum can also be simulated by methods similar to Monte-Carlo and molecular dynamics.

In order to calculate the Lamb shift, it is necessary to use:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta_D(\underline{r}) \quad -(8)$$

where $\delta_D(\underline{r})$ is the Dirac delta function. If the Dirac delta function is not used, then:

$$\nabla^2\left(\frac{1}{r}\right) = \nabla^4\left(\frac{1}{r}\right) = \nabla^6\left(\frac{1}{r}\right) = 0 \quad -(9)$$

and there is no vacuum correction to the Coulomb potential energy

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(10)$$

or the gravitational potential energy

$$U = -\frac{mMG}{r} \quad -(11)$$

However, if the Dirac delta function is used:

$$\nabla^2 U = -\frac{e^2}{4\pi\epsilon_0}\nabla^2\left(\frac{1}{r}\right) = \frac{e^2}{\epsilon_0}\delta_D(\underline{r}) \quad -(12)$$

$$\nabla^4 U = \frac{e^2}{\epsilon_0}\nabla^2\delta_D(\underline{r}) \quad -(13)$$

$$\nabla^6 U = \frac{e^2}{\epsilon_0}\nabla^4\delta_D(\underline{r}) \quad -(14)$$

In contemporary mathematics there are many definitions of the Dirac delta function, and each one of these can be used to work out $\nabla^2\delta(\underline{r})$ and $\nabla^4\delta(\underline{r})$.

The first order correction is:

$$\langle\Delta U\rangle = \frac{e^2}{2\epsilon_0}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\,\delta_D(\underline{r}) \quad -(15).$$

for the Coulombic potential energy and

$$\langle\Delta U\rangle = 4\pi m M G\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\,\delta_D(\underline{r}) \quad -(16)$$

for the Newtonian potential energy.

So the total Coulombic potential energy in the presence of the vacuum is:

$$U = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{e^2}{2\epsilon_0}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\,\delta_D(\underline{r}) \quad -(17)$$

and the total Newtonian potential energy in the presence of the vacuum is:

$$U = -\frac{mMG}{r} + 4\pi m M G\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\,\delta_D(\underline{r}) \quad -(18)$$

4) On the quantum level the result (17) is verified with precision in the Lamb shift. On the classical and macroscopic levels, the vacuum corrects the Coulombic potential energy and the Newtonian potential energy.

There are two useful definitions of the Dirac delta function that can be used:

1)
$$\delta_D(r) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(n \cdot r) \quad - (19)$$

2)
$$\delta_D(r) = \lim_{n \to \infty} \frac{1}{2\pi} \left(\frac{\sin((n + 1/2) r)}{\sin(r/2)} \right). \quad - (20)$$

The function $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$ can be computer simulated by using ensembles of vacuum particles, and the vacuum corrections to the Coulomb and Newton potential energies can be evaluated.