15(7): Taylor Series Expansion.

The Taylor series is:

$$f(x) = f(a) + (x-a) f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \dots \quad -(1)$$

So:

$$f(x+\delta x) = f(x) + \delta x f'(x) + \frac{1}{2!}(\delta x)^2 f''(x) + \dots \quad -(2)$$

$$= f(x) + \delta x \frac{\partial f}{\partial x} + \frac{1}{2!}(\delta x)^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{3!}(\delta x)^3 \frac{\partial^3 f}{\partial x^3} + \frac{1}{4!}(\delta x)^4 \frac{\partial^4 f}{\partial x^4} + \dots$$

$$= f(x) + \left(\delta x \frac{\partial}{\partial x}\right) f + \frac{1}{2!}\left(\delta x^2 \frac{\partial}{\partial x}\frac{\partial}{\partial x}\right) f + \frac{1}{3!}\left(\delta x^3 \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right) f + \frac{1}{4!}\left(\delta x^4 \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right) f + \dots$$

In three dimensions:

$$f(\underline{r} + \delta\underline{r}) = f(\underline{r}) + (\delta\underline{r} \cdot \underline{\nabla}) f(\underline{r}) + \frac{1}{2!}(\delta\underline{r} \cdot \underline{\nabla})^2 f(\underline{r}) + \frac{1}{3!}(\delta\underline{r} \cdot \underline{\nabla})^3 f(\underline{r}) + \frac{1}{4!}(\delta\underline{r} \cdot \underline{\nabla})^4 f(\underline{r}) \quad -(3)$$

$$+ \dots$$

2) So:

$$f(\underline{r}+\delta\underline{r})-f(\underline{r}) = \left[\delta\underline{r}\cdot\underline{\nabla}+\frac{1}{2!}(\delta\underline{r}\cdot\underline{\nabla})^2+\frac{1}{3!}(\delta\underline{r}\cdot\underline{\nabla})^3 + \frac{1}{4!}(\delta\underline{r}\cdot\underline{\nabla})^4+\dots\right]f(\underline{r})$$

$$:= \Delta(f(\underline{r})). \quad -(4)$$

This is the change in any scalar function $f(\underline{r})$ due to the vacuum fluctuation $\delta\underline{r}$.

If the vacuum is considered to be isotropic:

$$\langle\delta\underline{r}\rangle = \underline{0} \quad -(5)$$

It follows that:

$$\langle\delta\underline{r}\cdot\underline{\nabla}\rangle = \langle\delta\underline{r}\rangle\cdot\underline{\nabla} = 0. \quad -(6)$$

In the isotropic vacuum:

$$\langle\delta X^2\rangle = \langle\delta Y^2\rangle = \langle\delta Z^2\rangle = \frac{1}{3}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle$$

$$= \frac{1}{3}\left(\langle\delta X^2+\delta Y^2+\delta Z^2\rangle\right) \quad -(7)$$

and

$$\langle\delta X\delta Z\rangle = \langle\delta X\delta Y\rangle = \langle\delta Y\delta Z\rangle = 0 \quad -(8)$$

so

$$\langle(\delta\underline{r}\cdot\underline{\nabla})^2\rangle = \frac{1}{3}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\nabla^2 \quad -(9)$$

It follows that:

$$\Delta f(\underline{r}) = \frac{1}{6}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\nabla^2 f(\underline{r}) + \dots \quad -(10)$$

3) Therefore in summary:

$$\langle (\delta \underline{r} \cdot \underline{\nabla})^2 \rangle = \left\langle \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right)^2 \right\rangle$$

$$= \Big\langle \delta x^2 \frac{\partial^2}{\partial x^2} + \delta y^2 \frac{\partial^2}{\partial y^2} + \delta z^2 \frac{\partial^2}{\partial z^2}$$

$$+ \delta x \frac{\partial}{\partial x} \delta y \frac{\partial}{\partial y} + \cdots \Big\rangle$$

$$= \langle \delta x^2 \rangle \frac{\partial^2}{\partial x^2} + \langle \delta y^2 \rangle \frac{\partial^2}{\partial y^2} + \langle \delta z^2 \rangle \frac{\partial^2}{\partial z^2} \quad -(11)$$

$$= \frac{1}{3} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad -(12)$$

For higher order terms:

$$\langle (\delta \underline{r} \cdot \underline{\nabla})^3 \rangle = \left\langle \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right)^3 \right\rangle$$

$$= \langle (\delta x)^3 \rangle \frac{\partial^3}{\partial x^3} + \cdots \quad -(13)$$

$$= 0$$

The reason is that $\langle \delta x \rangle = 0$, so all odd powers such as $\langle (\delta x)^3 \rangle$ are also zero.

Consider:

$$\langle (\delta \underline{r} \cdot \underline{\nabla})^4 \rangle = \left\langle \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right)^4 \right\rangle$$

$$= \left\langle (\delta x)^4 \frac{\partial^4}{\partial x^2} + \cdots \right\rangle \quad -(14)$$

Using:

$$\langle (\delta x)^2 \rangle = \langle (\delta y)^2 \rangle = \langle (\delta z)^2 \rangle$$
$$= \frac{1}{3} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(15)$$

it follows that:

$$\langle (\delta x)^4 \rangle = \langle (\delta y)^4 \rangle = \langle (\delta z)^4 \rangle$$
$$= \frac{1}{9} \langle \delta \underline{r} \cdot \delta \underline{r} \; \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(16)$$

Continuing in this way, eq. (10) is developed into:

$$\Delta f(r) = \frac{1}{3.2!} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 f(r)$$
$$+ \frac{1}{9.4!} \langle \delta \underline{r} \cdot \delta \underline{r} \; \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^4 f(r)$$
$$+ \frac{1}{81.6!} \langle \delta \underline{r} \cdot \delta \underline{r} \; \delta \underline{r} \cdot \delta \underline{r} \; \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^6 f(r)$$
$$+ \cdots \quad -(17)$$

which is a rapidly converging series.