3(b)(3): Initial Condition

In general the field equations show that:

$$\frac{\ddot{X}}{\ddot{Y}} = \frac{\kappa_X}{\kappa_Y} \quad -(1)$$

for any force equation. Therefore eq. (1) can be used as an initial condition:

$$\frac{\ddot{X}(0)}{\ddot{Y}(0)} = \frac{\kappa_X(0)}{\kappa_Y(0)} \quad -(2)$$

for the solution of the various force equations. The orbit will depend on the initial condition, and therefore will depend on $\kappa_X(0)$ and $\kappa_Y(0)$. Having chosen the initial $\ddot{X}(0)$ and $\ddot{Y}(0)$, the orbit can be computed.

For Newtonian dynamics and retrograde precession:

$$\frac{\ddot{X}(0)}{\ddot{Y}(0)} = \frac{X(0)}{Y(0)} = \frac{\kappa_X(0)}{\kappa_Y(0)} \quad -(3)$$

For forward precession:

$$\frac{\ddot{X}(0)}{\ddot{Y}(0)} = \left(\frac{\dot{X}(0)\dot{Y}(0)Y(0) + X(0)\dot{X}^2(0)}{c^2} - X(0)\right)\left(\frac{\dot{X}(0)\dot{Y}(0)X(0) + Y(0)\dot{Y}^2(0)}{c^2} - Y(0)\right)^{-1}$$

$$= \frac{\kappa_X(0)}{\kappa_Y(0)} \quad -(4)$$

For example, the static ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(5)$$

and at the perihelion:

$$\phi = 2\pi n, \; n = 0, 1, 2, \ldots \quad -(6)$$

so at the perihelion:

$$r = (x^2+y^2)^{1/2} = \frac{\alpha}{1+\epsilon} \quad -(7)$$

here the half right latitude α and eccentricity ϵ are known from astronomy. If the perihelion is chosen as the initial condition:

$$r(0) = \left(X(0)^2 + Y(0)^2\right)^{1/2} = \frac{\alpha}{1+\epsilon} \quad -(8)$$

so

$$\left(x(0)^2 + \left(\frac{\kappa_y(0)}{\kappa_x(0)}\right)^2 x(0)^2\right)^{1/2} = \frac{\alpha}{1+\epsilon} \quad -(9)$$

so

$$x(0) = \left(\frac{\alpha}{1+\epsilon}\right)\left(1+\left(\frac{\kappa_y(0)}{\kappa_x(0)}\right)^2\right)^{-1/2} \quad -(10)$$

At the initial position the equation of the ellipse is:

$$\frac{X^2(0)}{a^2} + \frac{Y(0)^2}{b^2} = 1 \quad -(11)$$

so

$$x^2(0)\left(\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{\kappa_y(0)}{\kappa_x(0)}\right)^2\right) = 1 \quad -(12)$$

and

$$x(0) = \left(\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{\kappa_y(0)}{\kappa_x(0)}\right)\right)^{-1/2} \quad -(13)$$

From eq. (10) and (13) it is clear that $x(0)$

$$\cancel{\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{\kappa_y(0)}{\kappa_x(0)}\right)^2 = \frac{1}{a^2} + \frac{1}{b^2}\left(\frac{\kappa_y(0)}{\kappa_x(0)}\right)^2 \quad -(14)}$$

) depends on $\kappa_y(0)$ and $\kappa_x(0)$.

In general for a Newtonian orbit:

$$\frac{\ddot{x}}{\ddot{y}} = \frac{x}{y} = \frac{\kappa_x}{\kappa_y} \quad -(14)$$

$$\ddot{x} = -mG\frac{x}{(x^2+y^2)^{3/2}} \quad -(15)$$

$$\ddot{y} = -mG\frac{y}{(x^2+y^2)^{3/2}} \quad -(16)$$

From eq. (14):

$$x\kappa_y - y\kappa_x = 0 \quad -(17)$$

and

$$\ddot{x}\kappa_y - \ddot{y}\kappa_x = 0 \quad -(18)$$

therefore

$$\left(\frac{\kappa_x}{\kappa_y}\right)\ddot{y} = -mG\frac{x}{(x^2+y^2)^{3/2}} \quad -(19)$$

and

$$\left(\frac{\kappa_y}{\kappa_x}\right)\ddot{x} = -mG\frac{y}{(x^2+y^2)^{3/2}} \quad -(20)$$

For retrograde precession:

$$\ddot{x} = -\frac{mG}{\gamma^3}\frac{x}{(x^2+y^2)^{3/2}} \quad -(21)$$

$$\ddot{y} = -\frac{mG}{\gamma^3}\frac{y}{(x^2+y^2)^{3/2}} \quad -(22)$$

and:

4) $$\frac{\ddot{X}(\text{precessing})}{\ddot{X}(\text{static})} = \gamma^3 = \left(1 - \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^{-3/2} \quad -(23)$$

$$= \frac{\ddot{Y}(\text{precessing})}{\ddot{Y}(\text{static})} \quad -(24)$$

so to maximise the difference between the precessing and static orbits:

$$v^2 = \frac{\dot{X}^2 + \dot{Y}^2}{c^2} \quad -(25)$$

must be maximised.

Retrograde precession again obeys eq. (14), but the individual spin connections are different.

The problem is best solved by regarding the general result (14) as an initial condition:

$$\frac{\ddot{X}(0)}{\ddot{Y}(0)} = \frac{X(0)}{Y(0)} = \frac{\kappa_X(0)}{\kappa_Y(0)} \quad -(26)$$

then solving eqs. (15) and (11) or (21) and (22) simultaneously. A similar procedure should be used for forward precession, in which case:

$$\frac{\ddot{X}(0)}{\ddot{Y}(0)} = \frac{\kappa_X(0)}{\kappa_Y(0)} \quad -(27)$$

but

$$\frac{\ddot{X}(0)}{\ddot{Y}(0)} \neq \frac{X(0)}{Y(0)} \quad -(28)$$