374(5): The General Planar Orbit of Fluid Gravitation.

In general, the force equation becomes:

$$\underline{F} = m\,\underline{a} = -\frac{mMG}{r^2} \quad -(1)$$

where:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(2)$$

and

$$\underline{v} = \underline{v}\left(r(t), \phi(t), t\right) \quad -(3)$$

is the velocity field. Therefore:

$$\boxed{\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{MG}{r^2}\,\underline{e}_r} \quad -(4)$$

where

$$\begin{aligned}\underline{v} &= v_r\,\underline{e}_r + v_\phi\,\underline{e}_\phi \\ &= \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\end{aligned} \quad -(5)$$

In plane polar coordinates:

$$\underline{v}\cdot\underline{\nabla} = v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} \quad -(6)$$

so:

$$\begin{aligned}(\underline{v}\cdot\underline{\nabla})\underline{v} = &\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi}\right)\underline{e}_r \\ &+ \left(v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi}\right)\underline{e}_\phi\end{aligned} \quad -(7)$$

From previous notes and papers (UFT349 – UFT351):

$$\frac{D\underline{v}}{Dt} = -\underline{\nabla} h - \underline{\nabla}\,\Phi + \underline{f}_{visc} \quad -(8)$$

which is the Navier Stokes equation for the velocity field:

$$\underline{v} = \frac{D\underline{R}}{Dt}\left(\underline{r}(t), \phi(t), t\right) \quad -(9)$$

If it is assumed that:

$$\underline{\nabla} h = \underline{0} \quad -(10)$$

and

$$\underline{f}_{visc} = \underline{0} \quad -(11)$$

then eq. (8) becomes the gravitational Navier Stokes equation (4) with:

$$\Phi = -\frac{mG}{r} \quad -(12)$$

is the gravitational potential.

Therefore:

$$-\underline{\nabla}\,\Phi = -\frac{mG}{r^2}\underline{e}_r \quad -(13)$$

Here,

$$x = \underline{1} + \frac{\partial R_r}{\partial r} \quad -(14)$$

If x depends on time, then as in the previous note:

$$\frac{d\underline{v}}{dt} = \left(\dot{r}\dot{x} + x\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left((1+x)\dot{r}\dot{\phi} + r\ddot{\phi}\right)\underline{e}_\phi$$

$$-(15)$$

It follows that:

$$\dot{r}\dot{x} + x\ddot{r} - r\dot{\phi}^2 + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} = -\frac{mG}{r^2}$$

$$-(16)$$

3)
$$(1+x)\dot{r}\dot{\phi} + r\ddot{\phi} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} = 0 \quad -(17)$$

In classical orbit ~~theory~~:

$$\frac{\partial v_r}{\partial r} = \frac{\partial v_r}{\partial \phi} = \frac{\partial v_\phi}{\partial r} = \frac{\partial v_\phi}{\partial \phi} = 0 \quad -(18)$$

because

$$\underline{v} = \underline{v}(t). \quad -(19)$$

The velocity field of fluid dynamics on the other hand is:

$$\underline{v} = v(r(t), \phi(t), t) \quad -(20)$$

so the velocity field is a function of $r(t)$ and $\phi(t)$ as well as t.

From UFT 363, the velocity field is:

$$\underline{v} = \frac{D\underline{R}}{Dt} = x\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(21)$$

Therefore

$$v_r = x\dot{r} \quad -(22)$$

$$v_\phi = r\dot{\phi} \quad -(23)$$

It follows that

$$\frac{\partial v_r}{\partial r} = \frac{\partial (x\dot{r})}{\partial r} = \dot{r}\frac{\partial x}{\partial r} + x\frac{\partial \dot{r}}{\partial r} \quad -(24)$$

Here:

$$\frac{\partial x}{\partial r} = \frac{\partial^2 R_r}{\partial r^2} \quad -(25)$$

and

$$\frac{\partial \dot{r}}{\partial r} = \frac{\partial \dot{r}}{\partial t}\frac{\partial t}{\partial r} = \frac{\ddot{r}}{\dot{r}} \quad -(26)$$

It follows that:

4)
$$v_r \frac{\partial v_r}{\partial r} = xr\left(\dot{r}\frac{\partial x}{\partial r} + x\frac{\partial \dot{r}}{\partial r}\right)$$
$$= xr\left(\dot{r}\frac{\partial x}{\partial r} + \frac{\ddot{r}}{\dot{r}}\right) \quad -(27)$$

Similarly,
$$\frac{\partial v_r}{\partial \phi} = \frac{\partial(x\dot{r})}{\partial \phi} = \dot{r}\frac{\partial x}{\partial \phi} + x\frac{\partial \dot{r}}{\partial \phi} \quad -(28)$$

where:
$$\frac{\partial \dot{r}}{\partial \phi} = \frac{\partial \dot{r}}{\partial t}\frac{\partial t}{\partial \phi} = \frac{\ddot{r}}{\dot{\phi}} \quad -(29)$$

So
$$\frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} = \dot{\phi}\left(\dot{r}\frac{\partial x}{\partial \phi} + x\frac{\ddot{r}}{\dot{\phi}}\right)$$
$$= x\ddot{r} + \dot{r}\dot{\phi}\frac{\partial x}{\partial \phi}. \quad -(30)$$

Thirdly:
$$v_r \frac{\partial v_\phi}{\partial r} = x\dot{r}\frac{\partial(r\dot{\phi})}{\partial r} = x\dot{r}\left(\dot{\phi} + r\frac{\partial \dot{\phi}}{\partial r}\right)$$
$$= x\dot{r}\dot{\phi} + x\dot{r}r\frac{\ddot{\phi}}{\dot{r}}$$
$$= x\left(\dot{r}\dot{\phi} + r\ddot{\phi}\right). \quad -(31)$$

$$-(32)$$

Fourthly:
$$\frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} = \dot{\phi}\frac{\partial(r\dot{\phi})}{\partial \phi} = \dot{\phi}\left(\dot{\phi}\frac{\partial r}{\partial \phi} + r\frac{\partial \dot{\phi}}{\partial \phi}\right)$$

where:
$$\frac{\partial r}{\partial \phi} = \frac{\partial r}{\partial t}\frac{\partial t}{\partial \phi} = \frac{\dot{r}}{\dot{\phi}} \quad -(33)$$

5) and
$$\frac{\partial \dot{\phi}}{\partial \phi} = \frac{\partial \dot{\phi}}{\partial t}\frac{\partial t}{\partial \phi} = \frac{\ddot{\phi}}{\dot{\phi}} \quad -(34)$$

So
$$\frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} = \dot{\phi}\,\dot{r} + r\ddot{\phi} \quad -(35)$$

It follows that:
$$r\dot{x} + x\ddot{r} - r\dot{\phi}^2 + xr\left(\dot{r}\frac{\partial x}{\partial r} + \frac{\ddot{r}}{\dot{r}}\right) + x\ddot{r} + \dot{r}\dot{\phi}\frac{\partial x}{\partial \phi}$$
$$= -\frac{mG}{r^2} \quad -(36)$$

and:
$$(1+x)\dot{r}\dot{\phi} + r\ddot{\phi} + (1+x)(\dot{r}\dot{\phi} + r\ddot{\phi}) = 0 \quad -(37)$$

In these equations:
$$x = 1 + \frac{\partial R_r}{\partial r} \quad -(38)$$
$$\dot{x} = \frac{d}{dt}\left(\frac{\partial R_r}{\partial r}\right) \quad -(39)$$
$$\frac{\partial x}{\partial r} = \frac{\partial^2 R_r}{\partial r^2} \quad -(40)$$
$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{\partial R_r}{\partial r}\right) \quad -(41)$$

where:
$$R_r = R_r(r(t)) \quad -(42)$$

It follows that:
$$\frac{\partial x}{\partial \phi} = 0 \quad -(43)$$