374(6): Continuity and Vorticity Equations

Consider the flow velocity $\underline{v}$ of spacetime. The continuity equation is:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad -(1)$$

where

$$V = \frac{1}{\rho} \quad -(2)$$

is the specific volume. It follows that:

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{V}\frac{dV}{dt} = -\frac{1}{\rho}\frac{d\rho}{dt} \quad -(3)$$

In plane polar coordinates (r, ϕ):

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} \quad -(4)$$

$$= \frac{1}{r}\left(v_r + r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi}$$

$$= \frac{1}{r} v_r + \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi}$$

$$= \frac{1}{V}\frac{\partial V}{\partial t}$$

It follows that:

$$\boxed{\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} = \frac{1}{V}\frac{\partial V}{\partial t} - \frac{1}{r} v_r} \quad -(5)$$

In the transition from fluid dynamics to classical dynamics:

$$\underline{v}(r(t), \phi(t), t) \longrightarrow \underline{v}(t) \quad -(6)$$

so

$$\frac{\partial v_r}{\partial r} \to 0 \quad -(7)$$

and

$$\frac{\partial v_\phi}{\partial \phi} \to 0, \quad -(8)$$

so from eq. (5):

$$\frac{1}{r} v_r \to \frac{1}{V}\frac{\partial V}{\partial t}. \quad -(9)$$

In the Newtonian limit:

$$v_r \to \dot{r} \quad -(10)$$

so

$$\boxed{\frac{\dot{r}}{r} \to \frac{1}{V}\frac{\partial V}{\partial t}} \quad -(11)$$

If the fluid is incompressible then:

$$\frac{\partial V}{\partial t} = 0 \quad -(12)$$

and

$$\frac{\dot{r}}{r} \longrightarrow 0 \quad -(13)$$

Eq. (13) corresponds to a circular orbit. The area of the orbit does not change, and this area is cross section through a constant orbital volume. The area of the orbit is πr^2 and its volume is $\frac{4}{3}\pi r^3$. In other types of orbit, $\dot{r}/r$ is not zero

3) so the specific volume $\bar{V}$ changes. This can be thought of as a compressible spacetime, or a compressible fluid.

In order to determine $\frac{\partial v_r}{\partial r}$, $\frac{\partial v_\phi}{\partial \phi}$, $\frac{\partial v_\phi}{\partial r}$ and $\partial v_r / \partial \phi$, other equations of fluid dynamics are needed. Thus far we have used:

$$\underline{v} = \frac{\partial \underline{R}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{R} \quad - (14)$$

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = - \frac{MG}{r^2} \underline{e}_r \quad - (15)$$

and

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad - (16)$$

where

$$\underline{R} = R(r(t), \phi(t), t) \quad - (17)$$

Eq. (15) is the Navier Stokes equation with the gravitational force:

$$\underline{F} = m \underline{a} = - \frac{mMG}{r^2} \underline{e}_r \quad - (18)$$

There is also the vorticity equation, which is derived by taking the curl of both sides of eq. (15):

$$\underline{\nabla} \times \left(\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} \right) = - MG \, \underline{\nabla} \times \left(\frac{1}{r^2} \underline{e}_r \right) \quad - (19)$$

In cylindrical coordinates, the curl is defined as follows:

$$\underline{\nabla} \times \underline{F} = \left(\frac{1}{r} \frac{\partial F_Z}{\partial \phi} - \frac{\partial F_\phi}{\partial Z} \right) \underline{e}_r + \left(\frac{\partial F_r}{\partial Z} - \frac{\partial F_Z}{\partial r} \right) \underline{e}_\phi + \frac{1}{r} \left(\frac{\partial (r F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi} \right) \underline{k} \quad -(20)$$

The gravitational force is defined as:

$$\underline{F} = F_r \underline{e}_r \quad -(21)$$

here

$$F_r = -\frac{mMG}{r^2}. \quad -(22)$$

So: $F_\phi = F_Z = 0$ and $\partial F_r / \partial Z = \partial F_r / \partial \phi = 0. \quad -(23)$

It follows that:

$$\underline{\nabla} \times \left(\frac{1}{r^2} \underline{e}_r \right) = \underline{0}. \quad -(23)$$

Therefore:

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times \left((\underline{v} \cdot \underline{\nabla}) \underline{v} \right) = 0 \quad -(24)$$

where the vorticity of spacetime is:

$$\underline{w} = \underline{\nabla} \times \underline{v} \quad -(25)$$

Now use the vector identities:

$$(\underline{v} \cdot \underline{\nabla}) \underline{v} = \underline{\nabla} \left(\frac{1}{2} \underline{v} \cdot \underline{v} \right) - \underline{v} \times \underline{w} \quad -(26)$$

and

$$\underline{\nabla} \times (\underline{v} \times \underline{w}) = -\underline{w} (\underline{\nabla} \cdot \underline{v}) + (\underline{w} \cdot \underline{\nabla}) \underline{v} - (\underline{v} \cdot \underline{\nabla}) \underline{w} \quad -(27)$$

to find that:

$$\frac{D\underline{w}}{Dt} = \frac{\partial \underline{w}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{w} = (\underline{w}\cdot\underline{\nabla})\underline{v} - \underline{w}(\underline{\nabla}\cdot\underline{v}) \quad -(28)$$

which is the vorticity equation of spacetime relevant to orbital dynamics.

The term $(\underline{w}\cdot\underline{\nabla})\underline{v}$ describes the stretching or tilting of spacetime due to flow velocity gradients, and the term $\underline{w}(\underline{\nabla}\cdot\underline{v})$ describes the stretching of vorticity due to flow compressibility. The complete equation (28) describes the conservation of angular momentum. The continuity equation describes the conservation of matter.

Defining the velocity field by:

$$\underline{v} = v_r \underline{e}_r + v_\phi \underline{e}_\phi \quad -(29)$$

the vorticity is:

$$\begin{aligned}\underline{w} &= \underline{\nabla} \times \underline{v} \\ &= \frac{1}{r}\left(\frac{\partial (r v_\phi)}{\partial r} - \frac{\partial v_r}{\partial \phi}\right)\underline{k} \quad -(30) \\ &= \left(\frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right)\underline{k}\end{aligned}$$

so it is defined in terms of $\partial v_\phi / \partial r$ and $\partial v_r / \partial \phi$. From eq. (5), the continuity equation is defined in terms of $\partial v_r / \partial r$ and $\partial v_\phi / \partial r$.

In the Newtonian limit the Navier-Stokes

6) equation (15) reduces to:

$$m\frac{d\underline{v}}{dt} = -\frac{mMG}{r^2}\underline{e}_r \quad -(31)$$

so the vorticity equation reduces to:

$$\frac{d\underline{w}}{dt} = \underline{0} \quad -(32)$$

i.e. the velocity is defined by:

$$\underline{\nabla} \times \frac{d\underline{v}}{dt} = \underline{0} \quad -(33)$$

where

$$\frac{d\underline{v}}{dt} = (\ddot{r} - r\dot{\phi}^2)\underline{e}_r + (2\dot{r}\dot{\phi} + r\ddot{\phi})\underline{e}_\theta \quad -(34)$$

$$= (\ddot{r} - r\dot{\phi}^2)\underline{e}_r$$

because

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(35)$$

Therefore:

$$\underline{\nabla} \times ((\ddot{r} - r\dot{\phi}^2)\underline{e}_r) = \underline{0} \quad -(36)$$

in the Newtonian limit. Eq. (36) follows from the fact that:

$$\underline{a} = a_r\underline{e}_r \quad -(37)$$

where

$$a_r = \ddot{r} - r\dot{\phi}^2 \quad -(38)$$

so

$$\underline{\nabla} \times \underline{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial Z}\right)\underline{e}_r + \left(\frac{\partial a_r}{\partial Z} - \frac{\partial a_z}{\partial r}\right)\underline{e}_\phi + \frac{1}{r}\left(\frac{\partial (r a_\phi)}{\partial r} - \frac{\partial a_r}{\partial \phi}\right)\underline{k} \quad -(39)$$

) However:

$$a_Z = a_\phi = 0 \quad -(40)$$

So

$$\underline{\nabla} \times \underline{a} = \frac{\partial a_r}{\partial Z}\underline{e}_\phi - \frac{1}{r}\frac{\partial a_r}{\partial \phi}\underline{k} \quad -(41)$$

It follows that:

$$\frac{\partial a_r}{\partial Z} = 0 \quad -(42)$$

and

$$\frac{\partial a_r}{\partial \phi} = 0 \quad -(43)$$

From eqs. (38) and (43):

$$\frac{\partial}{\partial \phi}\left(\ddot{r} - r\dot{\phi}^2\right) = 0 \quad -(44)$$

in the Newtonian limit.

Conclusion

The orbit is defined by the following set of equations:

$$\underline{v} = \frac{D\underline{R}}{Dt} \quad -(45)$$

$$\frac{D\underline{v}}{Dt} = -\frac{mG}{r^2}\underline{e}_r \quad -(46)$$

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{V}\frac{dV}{dt} \quad -(47)$$

$$\frac{D\underline{w}}{Dt} = (\underline{w} \cdot \underline{\nabla})\underline{v} - \underline{w}(\underline{\nabla} \cdot \underline{v}) \quad -(48)$$

$$\underline{w} = \underline{\nabla} \times \underline{v} \quad -(49)$$

and an equation of conservation of energy to be

) discussed in the next note. These can be put in the form of the Kambe field equations, which have the same structure as ECE2 gravitational field equations and ECE2 electromagnetic field equations. It is known that eqns. (45) to (49) produce a precessing orbit. This is consistent with the fact that ECE2 relativity produces a precessing orbit. The orbit depends on the fluid dynamics of spacetime itself.