14(7): The Lagrangian of Fluid Dynamics

Consider the vector field of position:

$$\underline{r} = \underline{r}(r(t), \phi(t), t) \quad -(1)$$

and velocity

$$\underline{v} = \underline{v}(r(t), \phi(t), t) \quad -(2)$$

in plane polar coordinates (r, ϕ). As in note ...(3), the time derivative of velocity is:

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \frac{dr}{dt}\frac{\partial \underline{v}}{\partial r} + \frac{d\phi}{dt}\frac{\partial \underline{v}}{\partial \phi} \quad -(3)$$

$$= \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} \right) \underline{e}_r$$

$$+ \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} \right) \underline{e}_\phi$$

$$= \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v}$$

where:

$$(\underline{v} \cdot \underline{\nabla})\underline{v} = \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} \right)\underline{e}_r + \left(v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} \right)\underline{e}_\phi \quad -(4)$$

and

$$\frac{\partial \underline{v}}{\partial t} = (\ddot{r} - r\dot{\phi}^2)\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\underline{e}_\phi$$

$$= \left(\frac{\partial v_r}{\partial t} - \frac{v_\phi^2}{r} \right) \underline{e}_r + \left(\frac{\partial v_\phi}{\partial t} + \frac{v_\phi v_r}{r} \right) \underline{e}_\phi \quad -(5)$$

in the Newtonian limit:

$$v_r = \dot{r}, \quad v_\phi = r\dot{\phi}. \quad -(6)$$

In the limit of single particle classical dynamics:

$$\underline{v}(r(t), \phi(t), t) \rightarrow \underline{v}(t) \quad -(7)$$

and

$$(\underline{v} \cdot \underline{\nabla}) \underline{v} \rightarrow \underline{0} \quad -(8)$$

so

$$\frac{d\underline{v}}{dt} \rightarrow \frac{\partial \underline{v}}{\partial t} \quad -(9)$$

The Navier Stokes equation for gravitation is:

$$m \frac{d\underline{v}}{dt} = -\frac{mMG}{r^2} \underline{e}_r \quad -(10)$$

where $d\underline{v}/dt$ is given by equation (3).

The Lagrangian for eq. (10) is:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{mMG}{|\underline{r}|} \quad -(4)$$

where

$$\underline{r} = \underline{r}(r(t), \phi(t), t) \quad -(5)$$

The Euler Lagrange equation needed for eq. (4) is:

3)

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}}\right) \quad -(6)$$

As required by Hamilton's Principle of Least Action:

$$\mathcal{L} = \mathcal{L}(\underline{r}, \underline{\dot{r}}) \quad -(7)$$

In eqs (4) to (7):

$$\underline{v} = \underline{\dot{r}} = \frac{d\underline{r}}{dt} = \frac{\partial \underline{r}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{r} \quad -(8)$$

Note carefully that the potential energy:

$$U = -\frac{mMG}{|\underline{r}|} \quad -(9)$$

is no longer central, in other words the potential energy can be a function of r, ϕ and t.

The ECE2 covariant (relativistic) version of eq. (4) is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{|\underline{r}|} \quad -(10)$$

$$= -mc^2\left(1 - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|}$$

So an entirely self consistent theory of orbital precession has been obtained.