372(5): The Complete Set of Equations for the Relativistic Plane Polar Coordinates.

In plane polar coordinates, the velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(1)$$

where $\underline{e}_r$ and $\underline{e}_\phi$ are the unit vectors of the plane polar system, (r, ϕ). Therefore:

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(2)$$

The Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

and the lagrangian is:

$$\mathcal{L} = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{1/2} - U \quad -(4)$$

where U is the potential energy. For a Coulomb potential:

$$U = \frac{-e^2}{4\pi\epsilon_0 r} \quad -(5)$$

where ϵ_0 is the vacuum permittivity. The proper Lagrange variables are r and ϕ and the Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(6)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{d\phi}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(7)$$

2) The gradient operator is:

$$\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} \quad -(8)$$

Quantization occurs via:

$$i\hbar \partial^\mu \phi = p^\mu \phi \quad -(9)$$

where ϕ is the wavefunction of the relativistic two dimensional problem being considered here. In eq. (9)

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right) \quad -(10)$$

and

$$p^\mu = \left(\frac{E}{c}, \underline{p} \right) \quad -(11)$$

Here E is the relativistic energy:

$$E = \gamma m c^2 \quad -(12)$$

and $\underline{p}$ is the relativistic momentum:

$$\underline{p} = \gamma m \underline{v} \quad -(13)$$

It follows that:

$$\underline{\nabla} \phi = \frac{\partial \phi}{\partial r} \underline{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \phi} \underline{e}_\phi \quad -(14)$$

$$i\hbar \frac{\partial \phi}{\partial t} = \gamma m c^2 \phi \quad -(15)$$

$$-i\hbar \underline{\nabla} \phi = \gamma m \underline{v} \phi \quad -(16)$$

and

$$-\hbar^2 \nabla^2 \phi = p^2 \phi \quad -(17)$$

3) i.e. :

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = -\frac{m^2}{\hbar^2}\gamma^2\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\psi \quad -(18)$$

From eq. (16) :

$$-i\hbar\frac{\partial \psi}{\partial r} = \gamma m \dot{r}\psi \quad -(19)$$

and

$$-i\frac{\hbar}{r}\frac{\partial \psi}{\partial \phi} = m r \dot{\phi}\gamma\psi \quad -(20)$$

Computation Scheme

1) Find r, ϕ, $\dot{r}$ and $\dot{\phi}$ from eqs. (6) and (7).
2) Find $\psi(r, \phi)$ from eqs. (19) and (20).
3) Find $\psi(t)$ from eq. (15).
4) Check the results using eq. (18).

The Hamiltonian is :

$$H = \gamma m c^2 + U \quad -(21)$$

which can be expressed as :

$$H = \frac{p^2}{(1+\gamma)m} + mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(22)$$

therefore the Hamiltonian operator is :

$$\hat{H}\psi = \left(-\frac{\hbar^2\nabla^2}{(1+\gamma)m} + mc^2 - \frac{e^2}{4\pi\epsilon_0 r}\right)\psi \quad -(23)$$

4) and the energy levels are given by:

$$\hat{H}\phi = E\phi \quad - (24)$$

and

$$E = \langle E \rangle = \int \phi^* \hat{H} \phi \, d\tau \quad -(25)$$

These energy levels show two dimensional fine structure.

For the H atom this exercise is repeated using the spherical polar coordinates to give the wave functions of the relativistic H atom and its energy levels, shown fine structure. That will be the subject of the next note.