372(4): Relativistic Orbits and Relativistic H Atom

1) First consider the relativistic lagrangian of ECE2 theory in plane polar coordinates (r, ϕ):

$$\mathcal{L} = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(1)$$

where

$$v^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \quad -(2)$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(3)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(4)$$

These can be solved for:

$$\frac{dr}{d\phi} = \frac{dr}{dt}\frac{dt}{d\phi} = \frac{\dot{r}}{\dot{\phi}} \quad -(5)$$

The orbit is then:

$$r = \int \frac{dr}{d\phi}\, d\phi \quad -(6)$$

and should be a precessing ellipse. The amount of precession can be calculated and compared with data from astronomy.

2) The three dimensional relativistic lagrangian is:

2)
$$\mathcal{L} = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(7)$$

where
$$v^2 = \dot{r}^2 + r^2\dot{\beta}^2 \quad -(8)$$

and
$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(9)$$

in the spherical polar coordinate system (r, θ, ϕ).
There are three Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(10)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(11)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(12)$$

These three equations can be solved for the orbit:
$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} \quad -(13)$$

and
$$r = \int \frac{dr}{d\phi}\, d\phi \quad -(14)$$

and the orbit:
$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \quad -(15)$$

So
$$r = \int \frac{dr}{d\theta}\, d\theta \quad -(16)$$

3) These should be processing orbits in ϕ and θ.

3) The lagrangian for the relativistic hydrogen atom is

$$\mathcal{L} = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} + \frac{e^2}{4\pi\epsilon_0 r} \quad -(17)$$

where v^2 is given by eq. (8). Eqs. (17), (10), (11) and (12) can be solved simultaneously to give the relativistic momentum squared:

$$p^2 = \gamma^2 m^2 v^2 \quad -(18)$$

where

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(19)$$

and where v^2 is given by eq. (8). The wave functions of the relativistic H atom are given by:

$$-\hbar^2 \nabla^2 \phi = p^2 \phi = \gamma^2 m^2 v^2 \phi \quad -(20)$$

The energy levels of the relativistic H atom are given by

$$E = \int \phi^* H \phi \, d\tau \quad -(21)$$

where

$$H = \gamma mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(22)$$

4)

Eq (20) can be written as:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\beta^2} = -\frac{1}{f^2}\left(\dot{r}^2 + r^2\dot{\beta}^2\right)\psi \quad -(23)$$

where $\dot{r}$ and $\dot{\beta}$ are calculated from the lagrangian (7) and the two Euler Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \quad -(24)$$

and

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\beta}} \quad -(25)$$

This calculation gives the wave function ψ of the relativistic H atom.

The hamiltonian operator from eq. (22) is:

$$\hat{H}\psi = \left(\left(p^2c^2 + m^2c^4\right)^{1/2} + U\right)\psi = E\psi \quad -(26)$$

So

$$H = \left(p^2c^2 + m^2c^4\right)^{1/2} + U \quad -(27)$$

i.e.

$$p^2c^2 + m^2c^4 = \left(H - \frac{e^2}{4\pi\epsilon_0 r}\right)^2 \quad -(28)$$

where H is a constant of motion:

$$\frac{dH}{dt} = 0 \quad -(29)$$

Eq. (27) can be written as:

5) It follows that

$$H = E + U \quad -(30)$$

where:

$$E = \gamma mc^2 = (p^2c^2 + m^2c^4)^{1/2} \quad -(31)$$

so:

$$H = (p^2c^2 + m^2c^4)^{1/2} + U \quad -(32)$$

i.e

$$(H - U)^2 = E^2 = p^2c^2 + m^2c^4 \quad -(33)$$

so

$$E - mc^2 = \frac{c^2p^2}{E + mc^2} \quad -(34)$$

and

$$H_0 = H - mc^2 = \frac{c^2p^2}{E + mc^2} + U \quad -(35)$$

In the non relativistic limit:

$$E = \gamma mc^2 \underset{\gamma \to 1}{\longrightarrow} mc^2 \quad -(36)$$

so

$$H_0 \rightarrow \frac{p^2}{2m} + U \quad -(37)$$

Q.E.D. Eq. (37) is the non relativistic hamiltonian.

In general:

$$H_0 = \frac{c^2p^2}{(1+\gamma)mc^2} + U \quad -(38)$$

is the relativistic hamiltonian corrected for rest energy. Therefore the relevant hamiltonian operator for eq. (21) is:

$$\hat{H}_0 = -\frac{\hbar^2 \nabla^2}{(1+\gamma)m} + U \quad -(39)$$

here
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(40)$$

and
$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(41)$$

The analytical result for the energy levels of the relativistic H atom is:

$$E = mc^2\left(1 + \frac{\alpha^2}{\left(n - j - 1/2 + \left(\left(j+\frac{1}{2}\right)^2 - \alpha^2\right)^{1/2}\right)^2}\right)^{-1/2} \quad -(42)$$

where n is the principal quantum number and:
$$j = \ell \pm 1/2 \quad -(43)$$

where ℓ is the angular momentum quantum number. Here
$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(44)$$

the fine structure constant.

Therefore the numerical method should give the well known result (42).