372(6) The Relativistic H Atom of ECE2.

The lagrangian and hamiltonian are:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(1)$$

$$H = \gamma mc^2 + U \quad -(2)$$

where:

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(3)$$

and

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

The velocity is: $-(5)$

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi$$

in the spherical polar coordinate system (r, θ, ϕ) with unit vectors $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$. The del operator is

$$\underline{\nabla}\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}\underline{e}_\phi \quad -(6)$$

Quantization takes place with:

$$i\hbar\frac{\partial\phi}{\partial t} = E\phi = \gamma mc^2\phi \quad -(7)$$

and

$$-i\hbar\underline{\nabla}\phi = \gamma m\underline{v}\phi \quad -(8)$$

From eq. (8):

2)

$$-i\hbar \frac{\partial \varphi}{\partial r} = \gamma m \dot{r} \varphi \quad -(9)$$

$$-\frac{i\hbar}{r} \frac{\partial \varphi}{\partial \theta} = \gamma m r \dot{\theta} \varphi \quad -(10)$$

$$-\frac{i\hbar}{r \sin\theta} \frac{\partial \varphi}{\partial \phi} = \gamma m r \dot{\phi} \sin\theta \, \varphi \quad -(11)$$

Therefore the wave function $\varphi(r, \theta, \phi)$ can be found from simultaneous solution of Eqs. (9) to (11), in which

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad \rightarrow (12)$$

$$= \left(1 - \frac{\dot{r}^2 + r^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)}{c^2}\right)^{-1/2}$$

The Lagrangian is:

$$\mathcal{L} = -mc^2\left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\right)\right)^{1/2} + \frac{e^2}{4\pi\epsilon_0 r} \quad -(13)$$

and the Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(14)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(15)$$

and
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(16)$$

From eqs. (13) to (16), $\dot{r}$, $\dot{\theta}$ and $\dot{\phi}$ may be found, and knowing these, the wave function $\psi(r, \theta, \phi)$ may be found from eqs. (9) to (11).

Knowing $\psi(r, \theta, \phi)$, the energy levels E may be found from Eq. (7).

Usually the energy levels are found from:
$$\hat{H}\psi = E\psi \quad -(17)$$
where $\hat{H}$ is the hamiltonian operator. The classical hamiltonian H may be found from Eq. (2), which can be written as
$$H = \frac{p^2}{(1+\gamma)m} + mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(18)$$

Quantization of eq. (18) takes place through:
$$-\hbar^2 \nabla^2 \psi = p^2 \psi \quad -(19)$$

so
$$-\frac{\hbar^2 \nabla^2 \psi}{(1+\gamma)m} + \left(mc^2 - \frac{e^2}{4\pi\epsilon_0 r}\right)\psi = E\psi \quad -(20)$$

and
$$E = \langle E \rangle = \int \psi^* \hat{H} \psi \, d\tau \quad -(21)$$

4) Here:

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2} \quad -(22)$$

and

$$p^2 = \gamma^2 m^2 \left(\dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\right) \quad -(23)$$

Clearly, calculation of the energy levels is simpler with equation (7) than with the usual method (17) to (23).

Note that this method is completely new. The usual method is to set up and solve the Dirac equation using the Dirac spinors and gamma matrices. In ECE the Dirac equation has been developed into the fermion equation. This new method is developed in O(3) space, but can also be developed in SU(2) space