367(2): Euler Equations of Fluid Dynamics

Consider the position vector:

$$\underline{r} = r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3 \quad -(1)$$

Then as in Marion and Thornton problem 9.1:

$$\frac{D\underline{r}}{Dt} = \frac{d\underline{r}}{dt} + r_1 \dot{\underline{e}}_1 + r_2 \dot{\underline{e}}_2 + r_3 \dot{\underline{e}}_3 \quad -(2)$$

where:

$$\frac{d\underline{e}_1}{dt} = \omega_3 \underline{e}_2 - \omega_2 \underline{e}_3 \quad -(3)$$

$$\frac{d\underline{e}_2}{dt} = -\omega_3 \underline{e}_1 + \omega_1 \underline{e}_3 \quad -(4)$$

$$\frac{d\underline{e}_3}{dt} = \omega_2 \underline{e}_1 - \omega_1 \underline{e}_2 \quad -(5)$$

so

$$\frac{D\underline{r}}{Dt} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad -(6)$$

where the angular velocity is:

$$\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(7)$$

This analysis applies to any vector, if this is the angular momentum $\underline{L}$ then:

$$\frac{D\underline{L}}{Dt} = \frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L} \quad -(8)$$

$$= \frac{d}{dt}\left(L_1 \underline{e}_1 + L_2 \underline{e}_2 + L_3 \underline{e}_3\right)$$

2) These are the Euler equations.

Therefore the convective derivative is:

$$\left(\frac{D\underline{L}}{dt}\right)_{\text{convective}} = \frac{d}{dt}\left(L_1 \underline{e}_1 + L_2 \underline{e}_2 + L_3 \underline{e}_3\right) + \left(\underline{v} \cdot \underline{\nabla}\right)\underline{L} \quad -(9)$$

$$= \frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L} + \left(\underline{v} \cdot \underline{\nabla}\right)\underline{L}$$

i.e.:

$$\frac{D}{dt}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} + \begin{bmatrix} \omega_2 L_3 - \omega_3 L_2 \\ \omega_3 L_1 - \omega_1 L_3 \\ \omega_1 L_2 - \omega_2 L_1 \end{bmatrix} + \begin{bmatrix} \left(\left(\underline{v} \cdot \underline{\nabla}\right)\underline{L}\right)_1 \\ \left(\left(\underline{v} \cdot \underline{\nabla}\right)\underline{L}\right)_2 \\ \left(\left(\underline{v} \cdot \underline{\nabla}\right)\underline{L}\right)_3 \end{bmatrix} \quad -(10)$$

where:

$$\left(\underline{v} \cdot \underline{\nabla}\right)\underline{L} = \left(v_1 \frac{\partial}{\partial r_1} + v_2 \frac{\partial}{\partial r_2} + v_3 \frac{\partial}{\partial r_3}\right)\left(L_1 \underline{e}_1 + L_2 \underline{e}_2 + L_3 \underline{e}_3\right) \quad -(11)$$

New torques and new precessions are introduced by fluid dynamics.