347(6): The Lagrangian Analysis of the Motion of a Gyro

Ref. Marion and Thornton, Section 10.10.

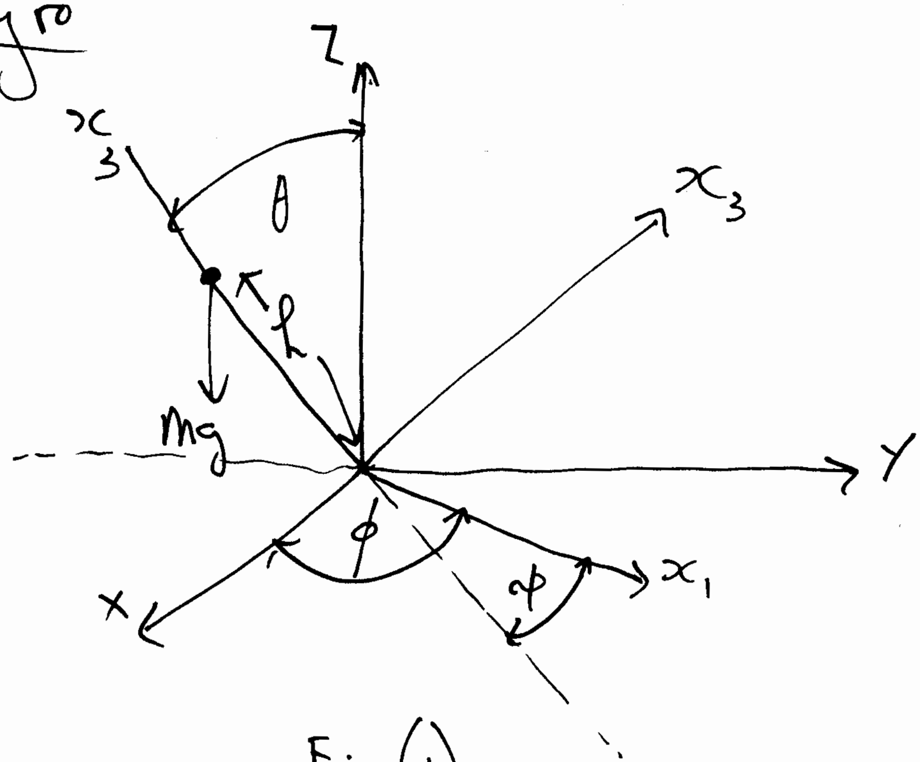

Fig (1).

Consider the gyro to be a symmetric top with principal moments of inertia:

$$I_{12} = I_1 = I_2 \quad -(1)$$

and I_3. Here (θ, ϕ, ψ) are the Euler angles.

The kinetic energy is:

$$T = \frac{1}{2} I_{12} \left(\omega_1^2 + \omega_2^2 \right) + \frac{1}{2} I_3 \omega_3^2 \quad -(2)$$

where:

$$\omega_1^2 + \omega_2^2 = \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \quad -(3)$$

$$\omega_3^2 = \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 \quad -(4)$$

Therefore

$$T = \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 \quad -(5)$$

2) The potential energy is:

$$U = mgh\cos\theta \quad -(6)$$

so the lagrangian is

$$\mathcal{L} = T - U \quad -(7)$$

The angular momenta:

$$L_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(8)$$

and

$$L_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad -(9)$$

are constants of motion. The lagrangian is:

$$\mathcal{L} = \frac{1}{2} I_{12}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - mgh\cos\theta \quad -(10)$$

In the Laithwaite configuration:

$$\theta = \frac{\pi}{2} \quad -(11)$$

so

$$\mathcal{L} = \frac{1}{2} I_{12}\left(\dot{\phi}^2 + \dot{\theta}^2\right) + \frac{1}{2} I_3 \dot{\psi}^2 \quad -(12)$$

It follows that:

$$L_\phi = I_{12}\dot{\phi} = \text{constant} \quad -(13)$$

and

$$L_\psi = I_3\dot{\psi} = \text{constant} \quad -(14)$$

Therefore:

3)

$$\boxed{\phi = \frac{L_\phi}{I_{12}} t} \quad - (15)$$

and

$$\boxed{\psi = \frac{L_\psi}{I_3} t} \quad - (16)$$

the angles ϕ and ψ increase linearly with time. The Hamiltonian in the Laithwaite configuration is:

$$H = \frac{1}{2} I_{12} (\dot{\phi}^2 + \dot{\theta}^2) + \frac{1}{2} I_3 \dot{\psi}^2$$

$$= \text{constant} \quad - (17)$$

so

$$\dot{\theta}^2 = \frac{2}{I_{12}} \left(H - \frac{1}{2} I_3 \dot{\psi}^2 - \frac{1}{2} I_{12} \dot{\phi}^2 \right)$$

$$= \text{constant} \quad - (18)$$

and

$$\boxed{\theta = \left[\frac{2}{I_{12}} \left(H - \frac{1}{2} I_3 \dot{\psi}^2 - \frac{1}{2} I_{12} \dot{\phi}^2 \right) \right]^{1/2} t \quad - (19)}$$

and θ also increases linearly with time.

The initial angular velocities in the Laithwaite experiment are

$$\omega_{30} = \dot{\theta}_0 \quad - (20)$$

and

$$\omega_{10} = \omega_{20} = 0 \quad - (21)$$

so the gyro starts to spin around the moving frame axes 1 and 2 as well as spinning in the moving frame axis 3, the initial axis of spin.

The forces on the gyro are governed by the equation:

$$\underline{F} = m\left(\frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v}\right)_{123} = -mg\,\underline{k} \quad -(22)$$

where

$$\frac{d\underline{v}}{dt} = \left(\frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}\right)_{123} \quad -(23)$$

where the subscript 123 denotes the moving frame (1, 2, 3) of Figure (1) defined by the principal moments of inertia of the gyroscope. So:

$$\underline{F} = m\left(\frac{d}{dt}\left(\frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}\right) + \underline{\omega} \times \left(\frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}\right)\right)_{123}$$

$$= -mg\,\underline{k} \quad -(24)$$

The force due to gravitation:

$$\underline{F} = -mg\,\underline{k} \quad -(25)$$

is balanced by the moving frame forces on the left hand side of eq. (24).

If there is no angular velocity $\underline{\omega}$

) then $m\left(\frac{d^2 \underline{r}}{dt^2}\right)_{123} = -mg\,\underline{k}$ – (26)

and there is nothing to counter-balance the force of gravitation. The non spinning gyro falls over.

In general:

$$F_Z^2 = F_1^2 + F_2^2 + F_3^2 \quad -(27)$$

$$= m^2 g^2$$

where:

$$F_1 = m\left(\frac{dv_1}{dt} + (\omega_2 v_3 - \omega_3 v_2)\right) \quad -(28)$$

$$F_2 = m\left(\frac{dv_2}{dt} + (\omega_3 v_1 - \omega_1 v_3)\right) \quad -(29)$$

$$F_3 = m\left(\frac{dv_3}{dt} + (\omega_1 v_2 - \omega_2 v_1)\right) \quad -(30)$$

and

$$v_1 = \frac{dr_1}{dt} + (\omega_2 r_3 - \omega_3 r_2) \quad -(31)$$

$$v_2 = \frac{dr_2}{dt} + (\omega_3 r_1 - \omega_1 r_3) \quad -(32)$$

$$v_3 = \frac{dr_3}{dt} + (\omega_1 r_2 - \omega_2 r_1) \quad -(33)$$

It is clear that:

$$F_Z^2 - (F_1^2 + F_2^2 + F_3^2) = 0 \quad -(34)$$

so there is no net counter gravitational force.

Finally, eq. (34) can be expressed as:

$$\underline{F} = m\left(\frac{d^2\underline{r}}{dt^2} + 2\underline{\omega}\times\frac{d\underline{r}}{dt} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\underline{\omega}}{dt}\times\underline{r}\right)_{123}$$

$$= -mg\,\underline{k} \quad -(35)$$

Therefore: $\quad -(36)$

$$m\frac{d^2\underline{r}}{dt^2} = -mg\,\underline{k} - m\left(2\underline{\omega}\times\frac{d\underline{r}}{dt} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\underline{\omega}}{dt}\times\underline{r}\right)_{123}$$

Here

$$\underline{F}_{Coriolis} = -2m\left(\underline{\omega}\times\frac{d\underline{r}}{dt}\right)_{123} \quad -(37)$$

$$\underline{F}_{centrifugal} = -m\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) - (38)$$

and

$$\underline{F}_3 = -m\frac{d\underline{\omega}}{dt}\times\underline{r} \quad -(39)$$

In planar orbital theory, eq. (36) reduces to the Leibnitz equation:

$$m\frac{d^2\underline{r}}{dt^2} = -mg\,\underline{} - m\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) - (40)$$

$$= -\frac{mMG}{r^2}\underline{e}_r + m\omega^2 r\,\underline{e}_r - (41)$$

where

$$\underline{r} = r\,\underline{e}_r \quad -(42)$$

) In self the gyro and orbital theory the gravitational force of attraction is counterbalanced exactly by a force of repulsion.

The description "moving frame" means that the axes of the frame are moving. The description "fixed frame" means that the axes of the frame are static.

So:

$$\underline{F} = m\left(\frac{d^2\underline{r}}{dt^2}\right)_{static} \qquad -(43)$$

$$= m\left(\frac{d^2\underline{r}}{dt^2} + 2\underline{\omega}\times\frac{d\underline{r}}{dt} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\underline{\omega}}{dt}\times\underline{r}\right)$$

Considering the F_3 component in eqs. (28) to (31), moving and assuming that F_1 and F_2 are zero:

$$F_3 = m\left(\frac{dV_3}{dt} + (\omega_1 V_2 - \omega_2 V_1)\right)$$

$$= m\left(\frac{d^2 r_3}{dt^2} + \frac{d}{dt}(\omega_1 r_2 - \omega_2 r_1) + \omega_1\left(\frac{dr_2}{dt} + (\omega_3 r_1 - \omega_1 r_3)\right) - \omega_2\left(\frac{dr_1}{dt} + (\omega_2 r_3 - \omega_3 r_2)\right)\right) \qquad -(44)$$

which is the 3 component of eq. (43).