47(7): Analysis of the Gyro in Spherical Polar Coordinates

In spherical polar coordinates:

$$\underline{r} = r\,\underline{e}_r \quad -(1)$$

and the velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\sin\theta\,\dot{\phi}\,\underline{e}_\phi \quad -(2)$$

("Vector Analysis Problem Solver" 21-25).

The acceleration is:

$$\underline{a} = a_r\,\underline{e}_r + a_\theta\,\underline{e}_\theta + a_\phi\,\underline{e}_\phi \quad -(3)$$

where:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 \quad -(4)$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 \quad -(5)$$

$$a_\phi = 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi} \quad -(6)$$

In a central force such as the gravitational force on a gyro:

$$\underline{F} = m\frac{d^2\underline{r}}{dt^2} = m\left(a_r\,\underline{e}_r + a_\theta\,\underline{e}_\theta + a_\phi\,\underline{e}_\phi\right) \quad -(7)$$

Therefore for the gyro:

$$-\frac{MG}{r^2} = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 \quad -(8)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 = 0 \quad -(9)$$

$$2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi} = 0 \quad -(10)$$

2) So the net force on the gyro is:

$$\underline{F} = m\frac{d^2\underline{r}}{dt^2} = -\frac{mMG}{r^2}\underline{e}_r$$
$$= m\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right)\underline{e}_r \qquad -(11)$$

i.e. the gyro generates a centrifugal force in three dimensions:

$$\underline{F}(\text{cent}) = mr\left(\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right)\underline{e}_r \qquad -(12)$$

Note that:

$$\frac{d^2\underline{r}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right)\underline{e}_r \qquad -(13)$$

so the presence of the centrifugal force does not change the equation:

$$\underline{F} = m\frac{d^2\underline{r}}{dt^2} = -\frac{mMG}{r^2}\underline{e}_r \qquad -(14)$$

and the gyro can never lift off the ground because it always obeys eq. (14).

Gyroscope dynamics are described by:

$$\ddot{r} = -\frac{MG}{r^2} + r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \qquad -(15)$$

which is the three dimensional equivalent of the Leibnitz orbital equation:

$$\ddot{r} = -\frac{mG}{r^2} + r\dot{\phi}^2 \quad -(16)$$

In spherical polar coordinates:

$$\frac{d\underline{e}_r}{dt} = \dot{\theta}\,\underline{e}_\theta + \sin\theta\,\dot{\phi}\,\underline{e}_\phi \quad -(17)$$

("Vector analysis Problem Solver" 21–24). So:

$$\frac{d\underline{e}_r}{dt}\cdot\frac{d\underline{e}_r}{dt} = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(18)$$

Fm eqs (15) and (18):

$$\boxed{\ddot{r} = -\frac{mG}{r^2} + r\,\frac{d\underline{e}_r}{dt}\cdot\frac{d\underline{e}_r}{dt}} \quad -(19)$$

where

$$\underline{r} = r\,\underline{e}_r \quad -(20)$$

Eq. (19) appears to be a new gyroscope equation.

In order to lift the gyroscope an additional force $\underline{F}_1$ is needed so that:

$$\underline{F} = \underline{F}_{grav} + \underline{F}_1 > \underline{0} \quad -(21)$$

In classical dynamics this lift can

be generated by a torque:

$$\underline{Tq} = \underline{r} \times \underline{F_1} \quad -(22)$$

If:

$$\underline{F_1} = F_1 \underline{e}_r \quad -(23)$$

experimentally then eq. (19) becomes:

$$\ddot{r} = -\frac{mG}{r^2} + r \frac{d\underline{e}_r}{dt} \cdot \frac{d\underline{e}_r}{dt} + F_1 \quad -(24)$$

otherwise $\underline{F_1}$ can be generated by a convective derivative:

$$\underline{F_1} = m(\underline{v} \cdot \underline{\nabla})\underline{v} \quad -(25)$$
