471(8): Moving Frame Force in Terms of Spherical Polar Coordinates

Let the moving frame be defined in terms of the principal moments of inertia, and be denoted frame (1, 2, 3). This is transformed to spherical polar coordinates as in Fig (1).

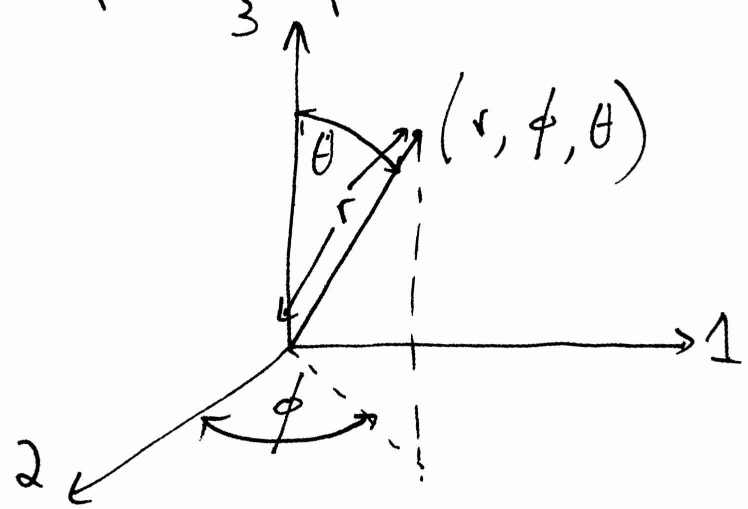

Fig (1)

So

$$r_1 = r \sin\theta \cos\phi \quad -(1)$$

$$r_2 = r \sin\theta \sin\phi \quad -(2)$$

$$r_3 = r \cos\theta \quad -(3)$$

The force in the moving frame is:

$$\underline{F} = F_1 \underline{e}_1 + F_2 \underline{e}_2 + F_3 \underline{e}_3 \quad -(4)$$

$$= F_r \underline{e}_r + F_\theta \underline{e}_\theta + F_\phi \underline{e}_\phi$$

Here:

$$F_1 = m\left(\frac{dV_1}{dt} + (\omega_2 V_3 - \omega_3 V_2)\right) \quad -(5)$$

$$F_2 = m\left(\frac{dV_2}{dt} + (\omega_3 V_1 - \omega_1 V_3)\right) \quad -(6)$$

$$F_3 = m\left(\frac{dV_3}{dt} + (\omega_1 V_2 - \omega_2 V_1)\right) \quad -(7)$$

2) as in note 347(6). In spherical polar coordinates:

$$F_r = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2) \quad -(8)$$

$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2) \quad -(9)$$

$$F_\phi = m(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi}) \quad -(10)$$

We have:

$$F_1^2 + F_2^2 + F_3^2 = F_r^2 + F_\theta^2 + F_\phi^2 \quad -(11)$$

$$= F_X^2 + F_Y^2 + F_Z^2$$

where the force in the lab. frame is:

$$\underline{F} = F_X\,\underline{i} + F_Y\,\underline{j} + F_Z\,\underline{k} \quad -(12)$$

The force in the lab frame is the force due to gravitation on the centre of mass of the gyro:

$$\underline{F} = mg\,\underline{k} = m\frac{d^2\underline{r}}{dt^2} \quad -(13)$$

Therefore the gyro can never lift off the ground without the application of an external force or torque.

In general:

$$\underline{F} = F_X\,\underline{i} + F_Y\,\underline{j} + F_Z\,\underline{k} \quad -(14)$$

$$= F_r\,\underline{e}_r + F_\theta\,\underline{e}_\theta + F_\phi\,\underline{e}_\phi$$

$$= F_1\,\underline{e}_1 + F_2\,\underline{e}_2 + F_3\,\underline{e}_3$$

) In the gyro:

$$\underline{F} = mg\,\underline{k} = F_r\,\underline{e}_r + F_\theta\,\underline{e}_\theta + F_\phi\,\underline{e}_\phi \quad -(15)$$

Note that if: $\dot{\theta} = \dot{\phi} = 0 \quad -(16)$

then:

$$\underline{F} = F_r\,\underline{e}_r = m\ddot{r}\,\underline{e}_r \quad -(17)$$

and

$$\underline{F} = mg\,\underline{k} = m\ddot{r}\,\underline{e}_r \quad -(18)$$

and the motion of the gyro is pure gravitational attraction between m and the mass of the earth M.
This happens when the gyro is not spun. Otherwise eq. (15) applies and the falling motion is counterbalanced by other terms.

In eq. (15):

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(19)$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$$

so

$$\underline{F} = mg\,\underline{k} = \left(F_r\cos\theta - F_\theta\sin\theta\right)\underline{k} \quad -(20)$$

$$-(21)$$

Therefore:

$$\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right)\cos\theta \qquad = g$$
$$-\left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)\sin\theta$$

Therefore:

$$\ddot{r}\cos\theta = g + \left(r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right)\cos\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)\sin\theta \quad -(22)$$

In the absence of spin:

$$\ddot{r}\cos\theta = g \quad -(23)$$

The acceleration due to gravity is:

$$g = -\frac{MG}{R^2} \quad -(24)$$

where M is the mass of the earth, R its radius, and G is Newton's constant. So in the gyroscope:

$$\boxed{\ddot{r}\cos\theta = -\frac{MG}{R^2} + \left(r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right)\cos\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)\sin\theta} \quad -(25)$$

and the force due to gravitation is counterbalanced by three dimensional centrifugal and Coriolis forces.

In the special case:

$$\theta = 0, \quad R = r \quad -(26)$$

we obtain the Leibnitz equation of 2-D orbits:

$$\ddot{r} = -\frac{MG}{r^2} + r\dot{\theta}^2 \quad -(27)$$

So the gyro is a type of 3D orbit.