67(3): Effect of Fluid Dynamics on the torque free motion of a Symmetric Top.

In fluid dynamics the torque in general is:

$$\underline{N} = \frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L} + (\underline{v} \cdot \underline{\nabla})\underline{L} \quad - (1)$$

and in classical dynamics the torque is:

$$\underline{N} = \frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L} \quad - (2)$$

in the notation of previous notes. Using principal moments of inertia I_i:

$$L_i = I_i \omega_i \quad - (3)$$

so eq. (1) becomes:

$$N_1 = I_1 \dot{\omega}_1 + (I_2 - I_3)\omega_2 \omega_3 + ((\underline{v} \cdot \underline{\nabla})\underline{L})_1$$

$$N_2 = I_2 \dot{\omega}_2 + (I_3 - I_1)\omega_3 \omega_1 + ((\underline{v} \cdot \underline{\nabla})\underline{L})_2$$

$$N_3 = I_3 \dot{\omega}_3 + (I_1 - I_2)\omega_1 \omega_2 + ((\underline{v} \cdot \underline{\nabla})\underline{L})_3$$

$$- (4)$$

Note that

$$(\underline{v} \cdot \underline{\nabla})\underline{L} = \left(\left(v_1 \frac{\partial}{\partial r_1} + v_2 \frac{\partial}{\partial r_2} + v_3 \frac{\partial}{\partial r_3} \right) (L_1 \underline{e}_1 + L_2 \underline{e}_2 + L_3 \underline{e}_3) \right) \quad - (5)$$

It follows that for torque free motion:

$$I_1 \dot{\omega}_1 + (I_2 - I_1)\omega_2 \omega_3 + x_1 = 0 \quad - (6)$$

$$I_2 \dot{\omega}_2 + (I_3 - I_1)\omega_3 \omega_1 + x_2 = 0 \quad -(7)$$

$$I_3 \dot{\omega}_3 + (I_1 - I_2)\omega_1 \omega_2 + x_3 = 0 \quad -(8)$$

here:

$$x_1 = I_1 \left(v_1 \frac{\partial \omega_1}{\partial r_1} + v_2 \frac{\partial \omega_1}{\partial r_2} + v_3 \frac{\partial \omega_1}{\partial r_3} \right) \quad -(9)$$

$$x_2 = I_2 \left(v_1 \frac{\partial \omega_2}{\partial r_1} + v_2 \frac{\partial \omega_2}{\partial r_2} + v_3 \frac{\partial \omega_2}{\partial r_3} \right) \quad -(10)$$

$$x_3 = I_3 \left(v_1 \frac{\partial \omega_3}{\partial r_1} + v_2 \frac{\partial \omega_3}{\partial r_2} + v_3 \frac{\partial \omega_3}{\partial r_3} \right) \quad -(11)$$

In classical dynamics:

$$x_1 = x_2 = x_3 = 0 \quad -(12)$$

The force free motion of a symmetric top:

$$I_1 = I_2 \neq I_3 \quad -(13)$$

in classical dynamics is described by:

$$(I_{12} - I_3)\omega_2 \omega_3 - I_{12} \dot{\omega}_1 = 0 \quad -(14)$$

$$(I_3 - I_{12})\omega_3 \omega_1 - I_{12} \dot{\omega}_2 = 0 \quad -(15)$$

$$I_3 \dot{\omega}_3 = 0 \quad -(16)$$

Therefore:

$$\omega_3 = \text{constant} \quad -(17)$$

and:

$$\dot{\omega}_1 = -\Omega\,\omega_2 \quad -(18)$$

$$\dot{\omega}_2 = \Omega\,\omega_1 \quad -(19)$$

where

$$\Omega = \left(\frac{I_3 - I_{12}}{I_{12}} \right) \omega_3 \quad -(20)$$

is the constant precession frequency.

In fluid dynamics, eq. (8) for a symmetric top is:

$$\dot{\omega}_3 = v_1 \frac{\partial \omega_3}{\partial r_1} + v_2 \frac{\partial \omega_3}{\partial r_2} + v_3 \frac{\partial \omega_3}{\partial r_3} \quad -(21)$$

so:

$$\omega_3 = \int \left(v_1 \frac{\partial \omega_3}{\partial r_1} + v_2 \frac{\partial \omega_3}{\partial r_2} + v_3 \frac{\partial \omega_3}{\partial r_3} \right) dt \quad -(22)$$

so ω_3 is changed, and it depends on the structure of the fluid vacuum or aether or spacetime. The equations of motion are:

$$\dot{\omega}_1 = -\Omega\,\omega_2 + x_1 \quad -(23)$$

$$\dot{\omega}_2 = \Omega\,\omega_1 + x_2 \quad -(24)$$

and the precession frequency is:

$$\boxed{\Omega = \left(\frac{I_3 - I_{12}}{I_{12}}\right)\int\left(v_1\frac{\partial\omega_3}{\partial r_1} + v_2\frac{\partial\omega_3}{\partial r_2} + v_3\frac{\partial\omega_3}{\partial r_3}\right)dt} \quad -(25)$$

The frequency is changed from that of classical dynamics, and the angular velocity ω_3 is no longer constant.

Various models of fluid flow can be used to evaluate eq. (25). The algebra simplifies if it is assumed that:

$$\frac{\partial\omega_3}{\partial r_1} = \frac{\partial\omega_3}{\partial t}\frac{\partial t}{\partial r_1} = \frac{1}{v_1}\frac{\partial\omega_3}{\partial t} \quad -(26)$$

$$\frac{\partial\omega_3}{\partial r_2} = \frac{\partial\omega_3}{\partial t}\frac{\partial t}{\partial r_2} = \frac{1}{v_2}\frac{\partial\omega_3}{\partial t} \quad -(27)$$

$$\frac{\partial\omega_3}{\partial r_3} = \frac{\partial\omega_3}{\partial t}\frac{\partial t}{\partial r_3} = \frac{1}{v_3}\frac{\partial\omega_3}{\partial t} \quad -(28)$$

in which case:

$$\Omega = 3\left(\frac{I_3 - I_{12}}{I_{12}}\right)\omega_3, \quad -(29)$$

however this is an over simplification.