363(2): Orbital Precession without Approximation

As in note 363(1):

$$V_r(P) = (1+\Omega^1{}_{01}) V_r(N) + \Omega^1{}_{02} V_\theta(N) \quad -(1)$$

$$V_\theta(P) = (1+\Omega^2{}_{02}) V_\theta(N) + \Omega^2{}_{01} V_r(N) \quad -(2)$$

Here P denotes precessing orbit and N the Newtonian orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(3)$$

It follows that:

$$V_\theta(N) = \frac{1}{A} V_r(N) \quad -(4)$$

and

$$V_r(N) = A\, V_\theta(N) \quad -(5)$$

where

$$A = \frac{\epsilon r}{\alpha}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(6)$$

so

$$V_R(P) = \left(1+\Omega^1{}_{01} + \frac{1}{A}\Omega^1{}_{02}\right) V_R(N) \quad -(7)$$

$$V_\theta(P) = \left(1+\Omega^2{}_{02} + A\,\Omega^2{}_{01}\right) V_\theta(N) \quad -(8)$$

and

$$\left(\frac{V_r}{V_\theta}\right)_P = B\left(\frac{V_r}{V_\theta}\right)_N \quad -(9)$$

where

$$B = \frac{1+\Omega^1{}_{01} + \frac{\Omega^1{}_{02}}{A}}{1+\Omega^2{}_{02} + A\,\Omega^2{}_{01}} \quad -(10)$$

a) In the Newtonian theory:

$$V_r(N) = \frac{dr}{dt}, \quad V_\theta(N) = r\frac{d\theta}{dt} \quad -(11)$$

So

$$\left(\frac{V_r}{V_\theta}\right)_N = \frac{1}{r}\frac{dr}{d\theta} \quad -(12)$$

$$= \frac{\epsilon r \sin\theta}{\alpha}$$

using

$$\frac{dr}{d\theta} = \frac{\epsilon r^2 \sin\theta}{\alpha} \quad -(13)$$

Now use:

$$\underline{v}(\underline{P}) = v_r(\underline{P})\,\underline{e}_r + v_\theta(\underline{P})\,\underline{e}_\theta \quad -(14)$$

$$= \left(\frac{dr}{dt}\right)_{\underline{P}} \underline{e}_r + r\left(\frac{d\theta}{dt}\right)_{\underline{P}} \underline{e}_\theta$$

So

$$\left(\frac{V_r}{V_\theta}\right)_{\underline{P}} = \frac{1}{r}\left(\frac{dr}{d\theta}\right)_{\underline{P}} \quad -(15)$$

From eq. (9)

$$\left(\frac{dr}{d\theta}\right)_{\underline{P}} = B\left(\frac{dr}{d\theta}\right)_N \quad -(16)$$

$$= \frac{B\epsilon r^2 \sin\theta}{\alpha}$$

Therefore the precessing orbit is:

3)

$$r = \frac{\epsilon}{\alpha} \int B r^2 \sin\theta \, d\theta \quad -(17)$$

$$= \frac{\epsilon}{\alpha} \int B \alpha^2 \frac{\sin\theta}{(1+\epsilon\cos\theta)^2} \, d\theta$$

Using:

$$A = \frac{\epsilon r}{\alpha} \sin\theta \quad -(18)$$

$$= \frac{\epsilon \sin\theta}{1+\epsilon\cos\theta}$$

it follows that:

$$B = \frac{1 + \Omega'_{01} + \Omega'_{02}\left(\frac{1+\epsilon\cos\theta}{\epsilon\sin\theta}\right)}{1 + \Omega^2_{02} + \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\Omega^2_{01}} \quad -(19)$$

So the general orbit can be worked out from eqs. (17) and (19).

The orbit can be modelled by choice of the spin connections, and this is true for any planar orbit.

From eq. (16) and (19):

$$\left(\frac{dr}{d\theta}\right)_P = \left(\frac{1 + \Omega'_{01} + \Omega'_{02}\left(\frac{1+\epsilon\cos\theta}{\epsilon\sin\theta}\right)}{1 + \Omega^2_{02} + \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\Omega^2_{01}}\right) \left(\frac{dr}{d\theta}\right)_N \quad -(20)$$

4) As shown in Note 363(1), if the precessing orbit is modelled by:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)}, \quad x = 1 + \frac{3MG}{c^2\alpha}, \quad -(21)$$

then

$$B = x. \quad -(22)$$

Therefore:

$$x = \frac{1+\Omega^1{}_{01}}{1+\Omega^2{}_{02}} \quad -(23)$$

and

$$\Omega^1{}_{02} = \Omega^2{}_{01} = 0 \quad -(24)$$

if x is a constant.

However, the result without approximation is eq. (20), which can be graphed for various values of $\Omega^1{}_{01}$, $\Omega^2{}_{02}$, $\Omega^1{}_{02}$ and $\Omega^2{}_{01}$.

From eqs. (1) and (2):

$$V_R(N) = \frac{(1+\Omega^2{}_{02})V_r(\underline{P}) - \Omega^1{}_{02}V_\theta(\underline{P})}{(1+\Omega^1{}_{01})(1+\Omega^2{}_{02}) - \Omega^1{}_{02}\Omega^2{}_{01}} \quad -(25)$$

$$V_\theta(N) = \frac{\Omega^2{}_{01}V_r(\underline{P}) - (1+\Omega^1{}_{01})V_\theta(\underline{P})}{(1+\Omega^1{}_{01})(1+\Omega^2{}_{02}) - \Omega^1{}_{02}\Omega^2{}_{01}} \quad -(26)$$

i.e. it follows that:

5)
$$V_r(N) = \frac{1}{A}\left(\left(1+\Omega^2_{02}\right)V_r(\underline{P}) - \Omega^1_{02}V_\theta(\underline{P})\right) \quad -(27)$$

$$V_\theta(N) = \frac{1}{A}\left(\left(\Omega^2_{01}V_r(\underline{P}) - \left(1+\Omega^1_{01}\right)V_\theta(\underline{P})\right)\right) \quad -(28)$$

where
$$A = \left(1+\Omega^1_{01}\right)\left(1+\Omega^2_{02}\right) - \Omega^1_{02}\Omega^2_{01} \quad -(29)$$

Using the result for Newtonian theory:

$$V_r^2 + V_\theta^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(30)$$

where a is the semi major axis, a relation can be obtained between $V_r(\underline{P})$ and $V_\theta(\underline{P})$:

$$\frac{1}{A^2}\left(\left(\left(1+\Omega^2_{02}\right)V_r(\underline{P}) - \Omega^1_{02}V_\theta(\underline{P})\right)^2 + \left(\Omega^2_{01}V_r(\underline{P}) - \left(1+\Omega^1_{01}\right)V_\theta(\underline{P})\right)^2\right) = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(31)$$

This can be worked out from computer algebra and used to find

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right)_{\underline{P}} = \left(\frac{V_r}{V_\theta}\right)_{\underline{P}} \quad -(32)$$
