363(5): Effect of the Spin Connection on the Orbital Binet Equation

From the previous note:

$$V_r = (1+\Omega^1_{01})\dot{r} + \Omega^1_{02}\omega r \quad -(1)$$

$$V_\theta = \omega r = \dot{\theta} r \quad -(2)$$

so the hamiltonian and lagrangian are:

$$H = \frac{1}{2} m (V_r^2 + V_\theta^2) + U \quad -(3)$$

and $$\mathcal{L} = \frac{1}{2} m (V_r^2 + V_\theta^2) - U \quad -(4)$$

From the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(5)$$

a force law can be found as follows. The hamiltonian is:

$$H = \frac{1}{2} m \left((1+\Omega^1_{01})^2 \dot{r}^2 + \Omega^{1\,2}_{02} r^2 \dot{\theta}^2 + 2\Omega^1_{02}(1+\Omega^1_{01})\dot{r}\dot{\theta} r + \dot{\theta}^2 r^2 \right) + U \quad -(6)$$

with a similar expression for the lagrangian.

From Eq. (6):

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m \left((1+\Omega^1_{01})^2 \dot{r} + \Omega^1_{02}(1+\Omega^1_{01})\dot{\theta} r \right) \quad -(7)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m \left((1+\Omega^1_{01})^2 \ddot{r} + \Omega^1_{02}(1+\Omega^1_{01})(\ddot{\theta} r + \dot{\theta}\dot{r}) \right) \quad -(8)$$

2) and:

$$\frac{\partial \mathcal{L}}{\partial r} = m\left(\left(1+\Omega'^{\,2}_{02}\right) r\dot{\theta}^2 + \Omega'_{02}\left(1+\Omega'_{01}\right)\dot{r}\dot{\theta}\right) - \frac{\partial U}{\partial r} \quad -(9)$$

It follows that the force law is:

$$F = m\left(\left(1+\Omega'_{01}\right)^2 \ddot{r} + \Omega'_{02}\left(1+\Omega'_{01}\right)\left(r\ddot{\theta}+\dot{\theta}\dot{r}\right) + \Omega'_{02}\left(1+\Omega'_{01}\right)\dot{r}\dot{\theta}\right. \quad -(10)$$

$$\left. -\left(1+\Omega'^{\,2}_{02}\right) r\dot{\theta}^2 - \Omega'_{02}\left(1+\Omega'_{01}\right)\dot{r}\dot{\theta}\right)$$

In comparison the force law for an elliptical orbit is

$$F = m\left(\ddot{r} - r\dot{\theta}^2\right) \quad -(11)$$

So it is clear that the orbit is changed by Ω'_{01} and Ω'_{02}.

In order purely to simplify the problem use:

$$\Omega'_{02} = \frac{1}{r}\frac{\partial R_r}{\partial \theta} \sim 0, \quad -(12)$$

so:

$$H = \frac{1}{2}m\left(\left(1+\Omega'_{01}\right)^2\dot{r}^2 + \dot{\theta}^2 r^2\right) + U \quad -(13)$$

$$\mathcal{L} = \frac{1}{2}m\left(\left(1+\Omega'_{01}\right)^2\dot{r}^2 + \dot{\theta}^2 r^2\right) - U \quad -(14)$$

From the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(15)$$

and

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad -(16)$$

) is the angular momentum of the system. From Eq. (15)

$$\frac{dL}{dt} = 0 \quad -(17)$$

so the angular momentum is conserved.

In the approximation (12), the force law (10) is:

$$F = m\left(\left(1+\Omega^1{}_{01}\right)^2 \ddot{r} - r\dot{\theta}^2\right) \quad -(18)$$

Using Binet's change of variable:

$$u = \frac{1}{r} \quad -(19)$$

it follows as in many UFT papers and notes that:

$$\ddot{r} = -\frac{L^2}{m^2} u^2 \frac{d^2 u}{d\theta^2}, \quad r\dot{\theta}^2 = \frac{L^2}{m^2} u^3 \quad -(20)$$

(Marion and Thornton p. 249). From eqs. (18) and (20):

$$F = -\frac{L^2}{m}\left(\left(1+\Omega^1{}_{01}\right)^2 u^2 \frac{d^2 u}{d\theta^2} + u^3\right) \quad -(21)$$

So:

$$\boxed{\left(1+\Omega^1{}_{01}\right)^2 \frac{d}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2} F(r)} \quad -(22)$$

This is the Binet equation of fluid dynamics. It reduces to the Binet equation of classical dynamics

$$\Omega^1{}_{01} \rightarrow 0 \quad -(23)$$

1) The Binet equation of classical dynamics allows the calculation of the force law for any orbit. If the orbit is a conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(24)$$

the force law is

$$F = -\frac{mMG}{r^2} \quad -(25)$$

For a conic section:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) \quad -(26)$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\theta \quad -(27)$$

so

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} = -\frac{mr^2}{L}F(r) \quad -(28)$$

It follows that

$$F(r) = -\frac{L^2}{m\alpha r^2} \quad -(29)$$

It is known that:

$$L^2 = m^2MG\alpha \quad -(30)$$

so

$$F = -\frac{mMG}{r^2} \quad -(31)$$

Q.E.D.

In the presence of Ω'_{01} in eq. (22) the following is true:

5)

$$-\frac{mr^2}{L^2}F(r) = -\frac{\epsilon}{\alpha}\left(1+\Omega'_{01}\right)^2\cos\theta + \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) \quad -(32)$$

$$= \frac{1}{\alpha} + \frac{\epsilon\cos\theta}{\alpha}\left(1 - \left(1+\Omega'_{01}\right)^2\right) \quad -(33)$$

From eq. (26):

$$\frac{\epsilon\cos\theta}{\alpha} = \frac{1}{r} - \frac{1}{\alpha} \quad -(34)$$

so

$$-\frac{mr^2}{L^2}F(r) = \frac{1}{\alpha} + \left(\frac{1}{r}-\frac{1}{\alpha}\right)\left(1-\left(1+\Omega'_{01}\right)^2\right)$$

$$= \frac{1}{\alpha}\left(1-\left(1-\left(1+\Omega'_{01}\right)^2\right)\right) + \frac{1}{r}\left(1-\left(1+\Omega'_{01}\right)^2\right) \quad -(35)$$

Denote

$$y = 1-\left(1+\Omega'_{01}\right)^2 \quad -(36)$$

then

$$\boxed{F(r) = -\frac{L^2}{mr^2}\left(\frac{1-y}{\alpha}+\frac{y}{r}\right)} \quad -(37)$$

In UFT193 it was shown that the force law for the precessing orbit:

6)

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(38)$$

is:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{x^2}{\alpha} + \frac{1}{r}\left(1-x^2\right)\right) \quad -(39)$$

so

$$y = 1 - x^2 \quad -(40)$$

and

$$\boxed{x = 1 + \Omega^1{}_{01}} \quad -(41)$$

Experimentally:

$$x = 1 + \frac{3MG}{\alpha c^2} \quad -(42).$$

so

$$\boxed{\Omega^1{}_{01} = \frac{\partial R_r}{\partial r} = \frac{3MG}{\alpha c^2}} \quad -(43)$$

Therefore orbital precession is due to the fluid dynamic function $\partial R_r / \partial r$, which can be thought of as the rate of displacement of the aether.