363(4): Some Considerations of Fundamentals in Eulerian Fluid Dynamics

In order to construct the hamiltonian and lagrangian of a new force law consider the fundamentals of Eulerian fluid dynamics, in which any vector field $\underline{F}$ is defined as:

$$\underline{F} = \underline{F}(\underline{r}, t) \quad -(1)$$

i.e. it is a function of position and time. Therefore the velocity field $\underline{v}(\underline{r}, t)$ is the rate of change of the position vector $\underline{r}$ of a fluid element:

$$\underline{v} = \frac{D\underline{r}}{Dt} = \frac{\partial \underline{r}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{r} \quad -(2)$$

In classical dynamics $\underline{v}$ is the velocity of a single particle and is defined by:

$$\underline{v} = \underline{v}(t) \quad -(3)$$

In general, if $\underline{F}$ is defined in the plane polar system by:

$$\underline{F} = F_r \underline{e}_r + F_\theta \underline{e}_\theta \quad -(4)$$

Then as in note 362(5): $\quad -(5)$

$$\frac{D}{Dt}\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} + \begin{bmatrix} \frac{\partial F_r}{\partial r} & \frac{1}{r}\frac{\partial F_r}{\partial \theta} \\ \frac{\partial F_\theta}{\partial r} & \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} \end{bmatrix}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix}$$

i.e.

$$\frac{D\underline{F}}{Dt} = \frac{\partial \underline{F}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{F} \quad -(6)$$

2) Note that the third term in eq. (5) contains the partial derivatives with respect to r and θ, but the second term on the right hand side of eq. (5') contains only

$$\omega = \dot{\theta} = \frac{d\theta}{dt} \quad -(7)$$

If

$$\underline{F} = \underline{F}(t) \quad -(8)$$

we obtain the result:

$$\frac{D}{Dt}\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} \quad -(9)$$

Therefore the classical dynamical definition (8) leads to:

$$\begin{bmatrix} \frac{\partial F_r}{\partial r} & \frac{1}{r}\frac{\partial F_r}{\partial \theta} \\ \frac{\partial F_\theta}{\partial r} & \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad -(10)$$

for the classical dynamics of particles.

In fluid dynamics eqn. (10) is not zero.

So eq. (10) results from:

$$\underline{F} = \underline{F}(t) \rightarrow \underline{F}(\underline{r}, t) \quad -(11)$$

Now consider the case when:

$$\underline{F} = \underline{r}. \quad -(12)$$

3) This requires that:

$$\underline{r} = \underline{r}(\underline{r}, t), \quad -(13)$$

which must be interpreted as:

$$\underline{R} = \underline{R}(\underline{r}, t) \quad -(14)$$

where $\underline{R}$ is the position vector of a fluid element. Therefore the velocity field in Euler's fluid dynamics is

$$\underline{v} = \frac{D\underline{R}}{Dt} = \frac{\partial \underline{R}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{R} \quad -(15)$$

i.e.:

$$\frac{D}{Dt}\begin{bmatrix} R_r \\ R_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} R_r \\ R_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} R_r \\ R_\theta \end{bmatrix} \quad -(16)$$

$$+ \begin{bmatrix} \frac{\partial R_r}{\partial r} & \frac{1}{r}\frac{\partial R_r}{\partial \theta} \\ \frac{\partial R_\theta}{\partial r} & \frac{1}{r}\frac{\partial R_\theta}{\partial \theta} \end{bmatrix}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix}$$

In plane polar coordinates:

$$\underline{R} = R\,\underline{e}_r \quad -(17)$$

so

$$R_r = R, \quad R_\theta = 0 \quad -(18)$$

The spin connections needed to define the velocity field are therefore given by:

f)

$$\begin{bmatrix} \Omega^1{}_{01} & \Omega^1{}_{02} \\ \Omega^2{}_{01} & \Omega^2{}_{02} \end{bmatrix} = \begin{bmatrix} \frac{\partial R_r}{\partial r} & \frac{1}{r}\frac{\partial R_r}{\partial \theta} \\ 0 & 0 \end{bmatrix} \quad -(19)$$

So

$$\Omega^1{}_{01} = \frac{\partial R_r}{\partial r}, \quad \Omega^1{}_{02} = \frac{1}{r}\frac{\partial R_r}{\partial \theta},$$

$$\Omega^2{}_{01} = 0 \; ; \; \Omega^2{}_{02} = 0 \quad -(20)$$

so the velocity components are

$$V_r = \left(1 + \Omega^1{}_{01}\right)\dot{r} + \Omega^1{}_{02}\,\omega r \quad -(21)$$

and

$$V_\theta = \omega r \quad -(22)$$

and the hamiltonian is:

$$H = \frac{1}{2} m \left(V_r^2 + V_\theta^2\right) + U \quad -(22)$$

The lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(V_r^2 + V_\theta^2\right) - U. \quad -(23)$$

Here U is the potential energy.

In orbital theory, $\Omega^1{}_{01}$ and $\Omega^1{}_{02}$ produce tiny corrections to an orbit, so:

$$\Omega^1{}_{01} \sim \Omega^1{}_{02} \ll 1. \quad -(24)$$

This is true in the solar system, but in galaxies

5) Ω'_{01} and Ω'_{02} can become large, and the orbit can be changed from a conic section to a hyperbolic spiral.

It follows that the velocity in plane polar coordinates can be defined by:

$$\underline{v} = \frac{D\underline{r}(t)}{Dt} = \frac{\partial \underline{r}(t)}{\partial t} + (\underline{v}\cdot\underline{\nabla})\,\underline{r}(t) \quad -(25)$$

which leads to:

$$\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} r(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} r(t) \\ 0 \end{bmatrix} \quad -(26)$$

i.e

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta . \quad -(27)$$

In classical dynamics:

$$\underline{R}(\underline{r}, t) = \underline{r}(t) \quad -(28)$$

These definitions allow the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(29)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(30)$$

to be evaluated with the lagrangian (23). This will be the subject of the next note.