356(4): Velocity Field of a Static Electric Field

In spherical polar coordinates (r, ϕ, θ), the static electric field in, for example, a circuit, is defined by:

$$\underline{E} = -\frac{e^2}{4\pi\epsilon_0 r^2}\,\underline{e}_r \quad -(1)$$

$$= x\left(\underline{v}\cdot\underline{\nabla}\right)\underline{v} \quad -(2)$$

in the notation of note 356(4).

The divergence of the velocity field is:

$$\underline{\nabla}\cdot\underline{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) \quad -(3)$$

where:

$$\underline{\nabla} = \nabla_r\,\underline{e}_r + \nabla_\phi\,\underline{e}_\phi + \nabla_\theta\,\underline{e}_\theta \quad -(4)$$

and

$$\underline{v} = v_r\,\underline{e}_r + v_\phi\,\underline{e}_\phi + v_\theta\,\underline{e}_\theta \quad -(5)$$

So:

$$\nabla_r v_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) = \frac{1}{r^2}\left(2 r v_r + r^2\frac{\partial v_r}{\partial r}\right) \quad -(6)$$

$$\nabla_\phi v_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi \quad -(7)$$

$$\nabla_\theta v_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) = \frac{1}{r}\left(\frac{\cos\theta}{\sin\theta} + 1\right)\frac{\partial}{\partial\theta}v_\theta \quad -(8)$$

It follows that:

$$\nabla_r = \frac{2}{r} + \frac{\partial}{\partial r} \quad -(9)$$

$$\nabla_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \quad -(10)$$

$$\nabla_\theta = \frac{1}{r}\left(1 + \frac{\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\theta} . \quad -(11)$$

Therefore:

$$\underline{v} \cdot \underline{\nabla} = v_r\left(\frac{2}{r} + \frac{\partial}{\partial r}\right) + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{v_\theta}{r}\left(1 + \frac{\cos\theta}{\sin\theta}\right)\frac{\partial}{\partial\theta} \quad -(12)$$

The spacetime velocity field set up by a static electric field in a circuit or any material is found by solving the following three differential equations simultaneously:

$$E_r = -\frac{e^2}{4\pi\epsilon_0 r^2} = x\left(\underline{v} \cdot \underline{\nabla}\right)v_r \quad -(13)$$

$$E_\phi = 0 = x\left(\underline{v} \cdot \underline{\nabla}\right)v_\phi \quad -(14)$$

$$E_\theta = 0 = x\left(\underline{v} \cdot \underline{\nabla}\right)v_\theta \quad -(15)$$

here $\underline{v} \cdot \underline{\nabla}$ is given by eq. (12).