33(3): Simplification of the Vorticity Equation.

In previous notes the vorticity equation was reduced to:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \mathbf{w} - \nabla h - \frac{1}{R} \nabla \times \mathbf{w} \quad -(1)$$

and was solved numerically in UFT352. The Kambe electric field is:

$$\mathbf{E}_F = -\nabla h - \frac{\partial \mathbf{v}}{\partial t} = (\mathbf{v} \cdot \nabla)\mathbf{v} \quad -(2)$$

so

$$\nabla h = -\frac{\partial \mathbf{v}}{\partial t} - (\mathbf{v} \cdot \nabla)\mathbf{v} \quad -(3)$$

From eqs. (1) and (3):

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \mathbf{w} + \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{R} \nabla \times \mathbf{w} \quad -(4)$$

so

$$\mathbf{v} \times \mathbf{w} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{R} \nabla \times \mathbf{w} = \mathbf{0} \quad -(5)$$

i.e.

$$\boxed{\nabla \times \mathbf{w} = R\left(\mathbf{v} \times \mathbf{w} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right)} \quad -(6)$$

in which

$$\mathbf{w} = \nabla \times \mathbf{v} \quad -(7)$$

is the vorticity. So:

$$\nabla \times (\nabla \times \mathbf{v}) = R\left(\mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) \quad -(8)$$

Now use the vector identity:

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2} \nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) \quad -(9)$$

From eqs. (8) and (9):

$$\boxed{\nabla \times (\nabla \times \mathbf{v}) = \frac{R}{2} \nabla v^2} \quad -(10)$$

Under the condition used by Kambe:

$$\underline{\nabla} h = \frac{1}{\rho} \underline{\nabla} p \quad -(11)$$

Eq. (10) is the simplest form of the vorticity equation. The original vorticity equation is:

$$\frac{\partial \underline{w}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{w}) + \frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} p + \frac{\mu}{\rho} \nabla^2 \underline{w}$$

$$-(12)$$

in which the Reynolds number is approximately:

$$R \sim \rho / \mu \quad -(13)$$

and by vector analysis:

$$\frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} p = - \underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} p \right)$$

$$= - \underline{\nabla} \times (\underline{\nabla} h) \quad -(14)$$

Therefore eq. (10) contains the same information as eq. (12), but is much simpler.

Now use the vector identity:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla} (\underline{\nabla} \cdot \underline{v}) - \nabla^2 \underline{v} \quad -(15)$$

so Eq (10) becomes:

$$\underline{\nabla} (\underline{\nabla} \cdot \underline{v}) - \nabla^2 \underline{v} = \frac{R}{2} \underline{\nabla} v^2 \quad -(16)$$

For incompressible, inviscid, flows:

3) $$\underline{\nabla} \cdot \underline{v} = 0, \quad \underline{R} = 0 \quad -(17)$$

So $$\nabla^2 \underline{v} = 0 \quad -(18)$$

Kambe's homogeneous field equation is:

$$\underline{\nabla} \times \underline{E}_F + \frac{\partial \underline{w}}{\partial t} = \underline{0} \quad -(19)$$

So $$\underline{\nabla} \times ((\underline{v} \cdot \underline{\nabla}) \underline{v}) + \frac{\partial}{\partial t} \underline{\nabla} \times \underline{v} = \underline{0} \quad -(20)$$

i.e. $$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = \underline{0} \quad -(21)$$

The original Navier Stokes equation is:

$$\frac{D\underline{v}}{Dt} = \underline{a} = -\frac{1}{\rho} \underline{\nabla} P - \underline{\nabla} \phi + \underline{f}_{visc} \quad -(22)$$

So Kambe's homogeneous field equation means that

$$\boxed{\underline{f}_{visc} = \frac{1}{\rho} \underline{\nabla} P + \underline{\nabla} \phi} \quad -(23)$$

From eqs. (9) and (21):

$$\frac{\partial \underline{v}}{\partial t} + \frac{1}{2} \underline{\nabla} v^2 - \underline{v} \times (\underline{\nabla} \times \underline{v}) = \underline{0} \quad -(24)$$

From eq. (10):

$$\underline{\nabla} v^2 = \frac{2}{R} \underline{v} \times (\underline{\nabla} \times \underline{v}) \quad -(25)$$

So

$$\frac{\partial \underline{v}}{\partial t} + \frac{1}{R}\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) - \underline{v} \times (\underline{\nabla} \times \underline{v}) = \underline{0} \quad -(26)$$

i.e.

$$\underline{v} \times (\underline{\nabla} \times \underline{v}) = \frac{\partial \underline{v}}{\partial t} + \frac{1}{R}\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) \quad -(27)$$

or:

$$\boxed{\frac{\partial \underline{v}}{\partial t} = \underline{v} \times \underline{w} - \frac{1}{R}\underline{\nabla} \times \underline{w}} \quad -(28)$$

Eq. (28) appears to be inconsistent with eq. (1). The source of the inconsistency is that:

$$\underline{\nabla} \times \underline{\nabla} h = \underline{0} \quad -(29)$$

in eq. (14).

So the simplest form of the vorticity equation (12) is eq. (28).

From eq. (16):

$$\nabla^2 \underline{v} = \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) - \frac{R}{2}\underline{\nabla} v^2 \quad -(30)$$

and from eq. (21):

$$\frac{\partial^2 \underline{v}}{\partial t^2} = -\frac{\partial}{\partial t}\left((\underline{v} \cdot \underline{\nabla})\underline{v}\right) \quad -(31)$$

If a_0 is a constant velocity such as the constant speed of sound then a d'Alembertian

5) can be defined as:

$$\square = \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \quad -(32)$$

So:

$$\square \underline{v} = -\frac{1}{a_0^2}\frac{\partial}{\partial t}\left(\left(\underline{v}\cdot\underline{\nabla}\right)\underline{v}\right) - \underline{\nabla}\left(\underline{\nabla}\cdot\underline{v}\right) + \frac{R}{2}\underline{\nabla}v^2 \quad -(33)$$

$$:= \kappa^2 \underline{v}$$

where κ is a wavenumber magnitude. Eq. (33) is a wave equation with the same format as the ECE2 wave equation.

In general, the speed of sound is variable.