353(5): Development of Wave Equations

Consider the definition of $\underline{E}_F$ and $\underline{B}_F$ in the generalized Kambe theory:

$$\underline{E}_F = -\underline{\nabla}\,\Phi - \frac{\partial \underline{v}}{\partial t} \quad - (1)$$

$$\underline{B}_F = \underline{w} = \underline{\nabla} \times \underline{v} \quad - (2)$$

where:

$$\underline{\nabla}\,\underline{\Phi} = \underline{\nabla}(h + \phi) - \underline{f}_{visc} \quad - (3)$$

and where the viscous force is defined by:

$$\underline{f}_{visc} = \mu \nabla^2 \underline{v} + (\mu + \mu')\,\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad - (4)$$

The generalized Kambe equations are then equivalent to the rigorous Navier Stokes equation and are:

$$\underline{\nabla}\cdot\underline{E}_F = q_F \quad - (5)$$

$$\underline{\nabla}\times\underline{E}_F + \frac{\partial \underline{B}_F}{\partial t} = \underline{0} \quad - (6)$$

$$\underline{\nabla}\cdot\underline{B}_F = 0 \quad - (7)$$

and

$$\underline{\nabla}\times\underline{B}_F - \frac{1}{a_0^2}\frac{\partial \underline{E}_F}{\partial t} = \frac{1}{a_0^2}\,\underline{J}_F \quad - (8)$$

where

$$q_F = \underline{\nabla}\cdot\left((\underline{v}\cdot\underline{\nabla})\underline{v}\right) \quad - (9)$$

and

$$\underline{J}_F = a_0^2\,\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) - \frac{\partial}{\partial t}\left((\underline{v}\cdot\underline{\nabla})\underline{v}\right) \quad - (10)$$

Eqs. (6) and (7) follow directly from eqs.

a) (1) and (2):

$$\underline{\nabla} \cdot \underline{B}_F = \underline{\nabla} \cdot \underline{\nabla} \times \underline{V} = 0 \quad - (11)$$

and

$$\underline{\nabla} \times \left(- \underline{\nabla} \Phi - \frac{\partial \underline{V}}{\partial t} \right) + \frac{\partial}{\partial t} \left(\underline{\nabla} \times \underline{V} \right) \quad - (12)$$

$$= - \underline{\nabla} \times \underline{\nabla} \Phi - \frac{\partial}{\partial t} \left(\underline{\nabla} \times \underline{V} \right) + \frac{\partial}{\partial t} \underline{\nabla} \times \underline{V} = \underline{0}$$

Eq. (5) is:

$$\underline{\nabla} \cdot \left(- \underline{\nabla} \Phi - \frac{\partial \underline{V}}{\partial t} \right) = q_F \quad - (13)$$

i.e.

$$\boxed{\nabla^2 \Phi + \frac{\partial}{\partial t} \left(\underline{\nabla} \cdot \underline{V} \right) = - q_F} \quad - (14)$$

This should be compared with Eq. (8) of Note 352(6):

$$\nabla^2 \phi_W + \frac{\partial}{\partial t} \left(\underline{\nabla} \cdot \underline{W} \right) = - \rho / \epsilon_0 \quad - (15)$$

Eq. (8) becomes:

$$\underline{\nabla} \times \left(\underline{\nabla} \times \underline{V} \right) - \frac{1}{a_0^2} \frac{\partial}{\partial t} \left(- \frac{\partial \underline{V}}{\partial t} - \underline{\nabla} \Phi \right) = \frac{\underline{J}_F}{a_0^2}$$

$$= - \nabla^2 \underline{V} + \underline{\nabla} \left(\underline{\nabla} \cdot \underline{V} \right) + \frac{1}{a_0^2} \frac{\partial^2 \underline{V}}{\partial t^2} + \frac{1}{a_0^2} \underline{\nabla} \Phi \quad - (16)$$

so

$$\boxed{\square \underline{V} + \underline{\nabla} \left(\underline{\nabla} \cdot \underline{V} + \frac{1}{a_0^2} \frac{\partial \Phi}{\partial t} \right) = \frac{1}{a_0^2} \underline{J}_F} \quad - (17)$$

here

$$\square := \frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad - (18)$$

3) Define

$$V^\mu := \left(\frac{\Phi}{a_0}, \underline{V} \right) \quad -(19)$$

and

$$\partial_\mu := \left(\frac{1}{a_0} \frac{\partial}{\partial t}, \underline{\nabla} \right) \quad -(20)$$

It follows that:

$$\partial_\mu V^\mu = \frac{1}{a_0^2} \frac{\partial \Phi}{\partial t} + \underline{\nabla} \cdot \underline{V} \quad -(21)$$

If the Lorenz gauge is assumed:

$$\partial_\mu V^\mu = 0 \quad -(22)$$

then Eq. (17) reduces to:

$$\Box \underline{V} = \frac{1}{a_0^2} \underline{J}_F \quad -(23)$$

Eqs. (17) and (23) have the structure of the ECE wave equation.

If the Lorenz gauge (22) is used in Eq (14), it follows that:

$$\boxed{\Box \Phi = q_F} \quad -(24)$$

i.e.

$$\left(\frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \Phi = q_F \quad -(25)$$

Now define:

4)

$$J^\mu = (a_0 q_{V_F}, \underline{J}_F) \quad - (26)$$

then

$$\boxed{\Box V^\mu = J^\mu} \quad - (27)$$

which can be regarded as the second order wave equation of fluid mechanics.

Eqs. (9) and (10) imply the continuity equation:

$$\frac{\partial q_{V_F}}{\partial t} + \underline{\nabla} \cdot \underline{J}_F = 0 \quad - (28)$$

i.e

$$\partial_\mu J^\mu = 0 \quad - (29)$$

Eq. (29) gives a justification for the Lorenz gauge (22) if:

$$J^\mu \,\mathcal{L}\, V^\mu \quad - (30)$$

i.e

$$\underline{J}_F \,\mathcal{L}\, \underline{V} \quad - (31)$$

and

$$q_{V_F} \,\mathcal{L}\, \Phi \quad - (32)$$

The units of Φ are $m^2 s^{-2}$ and those of q_{V_F} are s^{-2}. The units of $\underline{V}$ are $m s^{-1}$ and the units of $\underline{J}_F$ are $m s^{-3}$, so it follows that one possibility is:

$$\nabla^2 \Phi = q_{V_F} \quad - (33)$$

and

$$\frac{\partial^2 \underline{V}}{\partial t^2} = \underline{J}_F \quad - (34)$$

→ However, another possibility is:

$$\square \Phi = q_F \quad -(35)$$

which is eq. (25), and:

$$\square V^\mu = \frac{J^\mu}{a_0^2} \quad -(36)$$

which is eqn. (23).

Therefore eqns. (22), (24) and (31) are consistent with the continuity equation and Lorenz gauge, Q.E.D.

Units Check

$$\square \Phi = m^{-2} m^2 s^{-2} = s^{-2}, \; q_F = s^{-2} \checkmark$$

$$\square \underline{V} = m^{-2} m s^{-1} = m^{-1} s^{-1};$$

$$\frac{1}{a_0^2} \underline{J}_F = s^2 m^{-2} m s^{-3} = m^{-1} s^{-1} \checkmark$$

Conclusion: The wave equations are:

$$\boxed{\begin{aligned} \square \Phi &= q_F \quad -(37) \\ \square \underline{V} &= \frac{1}{a_0^2} \underline{J}_F \quad -(38) \end{aligned}}$$

These are rigorously compatible with the Navier-Stokes and vorticity equations.