353(2): Simplification of the Vorticity Equation

The vorticity equation is:

$$\frac{\partial \underline{w}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{w}) + \frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} P + \frac{\mu}{\rho} \nabla^2 \underline{w} \quad -(1)$$

where the Reynolds number is:

$$R \sim \frac{\rho}{\mu} \quad -(2)$$

Eq. (1) can be simplified by using:

$$\underline{w} = \underline{\nabla} \times \underline{v} \quad -(3)$$

and

$$\frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} P = - \underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} P \right) \quad -(4)$$

with:

$$\nabla^2 \underline{w} = \underline{\nabla} (\underline{\nabla} \cdot \underline{w}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{w}) \quad -(5)$$

so

$$\frac{\partial \underline{v}}{\partial t} = \underline{v} \times \underline{w} - \frac{1}{\rho} \underline{\nabla} P - \frac{1}{R} \underline{\nabla} \times \underline{w} \quad -(6)$$

Eq. (6) contains the same information as eq. (1).

In the Kambe theory:

$$\underline{\nabla} h = \frac{1}{\rho} \underline{\nabla} P \quad -(7)$$

so:

$$\frac{\partial \underline{v}}{\partial t} = \underline{v} \times \underline{w} - \underline{\nabla} h - \frac{1}{R} \underline{\nabla} \times \underline{w} \quad -(8)$$

) and:

$$\underline{E}_F = -\underline{\nabla} h - \frac{\partial \underline{v}}{\partial t} = (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(9)$$

So:

$$\underline{\nabla} h = \frac{1}{\rho}\underline{\nabla} p = -\frac{\partial \underline{v}}{\partial t} - (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(10)$$

Fm eqs. (8) and (10):

$$\boxed{\underline{v}\times\underline{w} - \frac{1}{R}\underline{\nabla}\times\underline{w} + (\underline{v}\cdot\underline{\nabla})\underline{v} = \underline{0}} \quad -(11)$$

i.e.

$$\underline{v}\times(\underline{\nabla}\times\underline{v}) - \frac{1}{R}\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) + (\underline{v}\cdot\underline{\nabla})\underline{v} = \underline{0} \quad -(12)$$

or

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) = R\left(\underline{v}\times(\underline{\nabla}\times\underline{v}) + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) \quad -(13)$$

Now use:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = \frac{1}{2}\underline{\nabla} v^2 - \underline{v}\times(\underline{\nabla}\times\underline{v}) \quad -(14)$$

So

$$\boxed{\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) = \frac{R}{2}\underline{\nabla} v^2} \quad -(15)$$

This is an equation for $\underline{v}$. To introduce time dependence use:

$$\underline{\nabla}\times\underline{E}_F + \frac{\partial \underline{w}}{\partial t} = \underline{0} \quad -(16)$$

So:

3)

$$\underline{\nabla} \times \left(\left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} \right) + \frac{\partial}{\partial t} \left(\underline{\nabla} \times \underline{v} \right) = \underline{0} \quad -(17)$$

so

$$\frac{\partial \underline{v}}{\partial t} + \left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} = \underline{0} \quad -(18)$$

i.e.

$$\frac{D\underline{v}}{Dt} = \underline{0} \quad -(19)$$

Eqs (15) and (19) completely describe the velocity field in terms of the Reynolds number.