353(1): Derivation of the Navier Stokes Equation and the Meaning of Terms in the Vorticity Equation

References: 1) Google: "Vorticity equation and Reynolds number"
2) "Vector Analysis Problem Solver" (VAPS)

Consider the force per unit volume in a fluid:

$$\underline{f} = \rho \, \underline{a} \quad -(1)$$

where ρ is the mass density in kilogram per metre cubed and $\underline{a}$ is the fluid acceleration (VAPS pp. 472 ff.). This is the force due to the external pressure P:

$$\underline{f}_P = - \underline{\nabla} P \quad -(2)$$

Units check: $\underline{f} = \text{kgm m}^{-3} \text{ m s}^{-2} = \text{kgm m}^{-2} \text{s}^{-2}$.

$-\underline{\nabla} P = \text{m}^{-1} \text{ kgm m}^{-2} = \text{kgm m}^{-3}$

It appears that the subject of hydrodynamics uses a special unit of pressure, or reduced unit. The S.I. unit of pressure is kgm m^{-2}, but P in Eq. (2) does not have these units.

Accepting this convention, the force due to the external pressure is counterbalanced by internal forces, described by the potential ϕ. Therefore the total force is:

$$\underline{f} = - \underline{\nabla} P - \rho \underline{\nabla} \phi \quad -(3)$$

$$= \rho \, \underline{a}$$

The acceleration is:

$$\underline{a} = \frac{D v}{D t} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} \quad -(4)$$

Therefore the Navier Stokes equation without

) viscosity effects is

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla} P - \rho\underline{\nabla}\phi} \quad -(5)$$

i.e. $$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{\underline{\nabla} P}{\rho} - \underline{\nabla}\phi \quad -(6)$$

In the presence of viscous force:

$$\boxed{\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla} P - \underline{\nabla}\phi + \underline{f}_{visc}} \quad -(7)$$

where $\underline{f}_{visc}$ is the viscous force.

The most general form of a second derivative that can occur in a vector equation is a linear combination of $\nabla^2\underline{v}$ and $\underline{\nabla}(\underline{\nabla}\cdot\underline{v})$. Therefore the viscous force is expressed (VAPS) as:

$$\underline{f}_{viscous} = \mu\nabla^2\underline{v} + (\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(8)$$

where μ and μ' are parameters to be determined.

Therefore the Navier Stokes equation is:

$$\rho\frac{D\underline{v}}{Dt} = \rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right)$$
$$= -\underline{\nabla} P - \rho\underline{\nabla}\phi + \mu\nabla^2\underline{v} + (\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(9)$$

i.e.

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla}P - \underline{\nabla}\phi + \left(\frac{\mu}{\rho}\right)\nabla^2\underline{v} + \frac{(\mu+\mu')}{\rho}\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(10)$$

In equation (10):

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = \frac{1}{2}\underline{\nabla}v^2 - \underline{v}\times(\underline{\nabla}\times\underline{v}) \quad -(11)$$

by a vector identity.

The vorticity is defined by:

$$\underline{w} = \underline{\nabla}\times\underline{v}. \quad -(12)$$

Therefore

$$\frac{\partial \underline{v}}{\partial t} + \frac{1}{2}\underline{\nabla}v^2 - \underline{v}\times\underline{w} = -\frac{1}{\rho}\underline{\nabla}\underline{P} - \underline{\nabla}\phi + \left(\frac{\mu}{\rho}\right)\nabla^2\underline{v} + \left(\frac{\mu+\mu'}{\rho}\right)\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(13)$$

and the Navier Stokes equation is:

$$\boxed{\frac{\partial \underline{v}}{\partial t} = \underline{v}\times\underline{w} + \frac{1}{2}\underline{\nabla}v^2 - \frac{1}{\rho}\underline{\nabla}P - \underline{\nabla}\phi + \left(\frac{\mu}{\rho}\right)\nabla^2\underline{v} + \frac{1}{\rho}(\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v})} \quad -(14)$$

which is the conservation of fluid linear momentum.

The conservation of fluid angular momentum is known as the vorticity equation and is the curl of eq. (14):

$$\underline{\nabla} \times \left(\frac{\partial \underline{v}}{\partial t}\right) = \frac{\partial}{\partial t}\underline{\nabla} \times \underline{v} = \frac{\partial \underline{w}}{\partial t}$$

$$= \underline{\nabla} \times (\underline{v} \times \underline{w}) + \frac{1}{2}\underline{\nabla} \times \underline{\nabla} v^2 - \underline{\nabla} \times \left(\frac{1}{\rho}\underline{\nabla} P\right)$$
$$- \underline{\nabla} \times (\underline{\nabla}\phi) + \left(\frac{\mu}{\rho}\right)\underline{\nabla} \times \nabla^2 \underline{v}$$
$$+ \frac{1}{\rho}(\mu + \mu')\underline{\nabla} \times \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) \quad -(15)$$

By vector analysis:

$$\frac{1}{2}\underline{\nabla} \times \underline{\nabla} v^2 = \underline{0}, \quad -(16)$$

$$-\underline{\nabla} \times \underline{\nabla}\phi = \underline{0}, \quad -(17)$$

and

$$\frac{1}{\rho}(\mu + \mu')\underline{\nabla} \times \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) = \underline{0} \quad -(18)$$

So the vorticity equation is:

$$\boxed{\frac{\partial \underline{w}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{w}) - \underline{\nabla} \times \left(\frac{1}{\rho}\underline{\nabla} P\right) + \frac{\mu}{\rho}\underline{\nabla} \times (\nabla^2 \underline{v})} \quad -(19)$$

The terms on the right hand side of eq. (19) can be developed with vector analysis as follows:

$$\underline{\nabla} \times (\underline{v} \times \underline{w}) = (\underline{w} \cdot \underline{\nabla})\underline{v} + \underline{v}(\underline{\nabla} \cdot \underline{w})$$
$$- \underline{w}(\underline{\nabla} \cdot \underline{v}) - (\underline{v} \cdot \underline{\nabla})\underline{w} \quad -(20)$$
$$= (\underline{w} \cdot \underline{\nabla})\underline{v} - \underline{w}(\underline{\nabla} \cdot \underline{v}) - (\underline{v} \cdot \underline{\nabla})\underline{w}$$

because:

$$\underline{\nabla} \cdot \underline{w} = \underline{\nabla} \cdot \underline{\nabla} \times \underline{v} = \underline{0} \quad -(21)$$

The second term is developed with the vector identity:

$$\underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} P \right) = \frac{1}{\rho} \underline{\nabla} \times \underline{\nabla} P + \underline{\nabla} \left(\frac{1}{\rho} \right) \times \underline{\nabla} P$$

$$= \underline{\nabla} \left(\frac{1}{\rho} \right) \times \underline{\nabla} P \quad -(22)$$

Now use VAPS Problem 7-10:

$$\underline{\nabla} \left(\frac{f}{g} \right) = \frac{1}{g^2} \left(g \underline{\nabla} f - f \underline{\nabla} g \right) \quad -(23)$$

where f and g are two scalars. When:

$$f = 1, \; g = \rho \quad -(24)$$

it is found that:

$$\underline{\nabla} \left(\frac{1}{\rho} \right) = -\frac{1}{\rho^2} \underline{\nabla} \rho. \quad -(25)$$

Therefore:

$$\underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} P \right) = -\frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} P \quad -(26)$$

<u>This</u> is <u>the baroclinic torque term</u>

The third term on the right hand side of Eq. (19) can be developed using:

$$\underline{\nabla} \times \left(\nabla^2 \underline{v} \right) = \nabla^2 \left(\underline{\nabla} \times \underline{v} \right) = \nabla^2 \underline{w} \quad -(27)$$

To prove eq. (27) note that:

b)

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \quad -(28)$$

and:

$$\underline{\nabla} \times \underline{v} = \underline{i}\left(\frac{\partial v_Z}{\partial Y} - \frac{\partial v_Y}{\partial Z}\right) + \cdots, \quad (29)$$

$$\nabla^2 \underline{v} = \nabla^2 v_X \,\underline{i} + \nabla^2 v_Y \,\underline{j} + \nabla^2 v_Z \,\underline{k}, \quad -(30)$$

So:

$$\underline{\nabla} \times \nabla^2 \underline{v} = \underline{i}\left(\frac{\partial}{\partial Y}\nabla^2 v_Z - \frac{\partial}{\partial Z}\nabla^2 v_Y\right) + \cdots \quad -(31)$$

and $$\nabla^2(\underline{\nabla} \times \underline{v}) = \nabla^2\left(\frac{\partial v_Z}{\partial Y} - \frac{\partial v_Y}{\partial Z}\right)\underline{i} + \cdots \quad (32)$$

Eqs. (31) and (32) are the same because of the commutativity:

$$\nabla^2 \frac{\partial v_Z}{\partial Y} = \frac{\partial}{\partial Y}\nabla^2 v_Z \quad -(33)$$

and so on, Q.E.D.

Therefore eq. (19) becomes: $\quad -(34)$

$$\boxed{\frac{\partial \underline{w}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{w}) + \frac{1}{\rho^2}\underline{\nabla}\rho \times \underline{\nabla} P + \frac{\mu}{\rho}\nabla^2 \underline{w}}$$

here:

$$\underline{\nabla} \times (\underline{v} \times \underline{w}) = (\underline{w} \cdot \underline{\nabla})\underline{v} - \underline{w}(\underline{\nabla} \cdot \underline{v}) - (\underline{v} \cdot \underline{\nabla})\underline{w}$$

$-(35)$

7) The coefficient of viscosity is:

$$\nu = \frac{\mu}{\rho} \quad -(36)$$

and the Reynolds number is:

$$R = \frac{\rho}{\mu} v d \quad -(37)$$

where v is the flow velocity and d a distance parameter.

If

$$v d \sim 1 \quad -(38)$$

the Reynolds number is:

$$R \sim \frac{\rho}{\mu} \quad -(39)$$

So eq. (34) becomes:

$$\boxed{\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{w} \times \underline{v}) = \frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} P + \frac{1}{R} \nabla^2 \underline{w} \quad -(40)}$$

This is the equation used in UFT 252, 251 and 249, but there is an extra baroclinic term, $\frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} P$, which originates in:

$$\underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} P \right) = -\frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} P \quad -(41)$$

Kambe used the equation:

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{w} \times \underline{v}) = \underline{0} \qquad -(42)$$

So considered a fluid of constant density with:

$$\underline{\nabla} \rho = \underline{0} \qquad -(43)$$

and high Reynolds number:

$$R \rightarrow \infty \qquad -(44)$$

Eq. (40) can be written as:

$$\frac{D\underline{w}}{Dt} = \frac{\partial \underline{w}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{w}$$

$$= (\underline{w} \cdot \underline{\nabla})\underline{v} - \underline{w}(\underline{\nabla} \cdot \underline{v}) + \frac{1}{\rho^2} \underline{\nabla}\rho \times \underline{\nabla} \underline{p} + \frac{1}{R} \nabla^2 \underline{w} \qquad -(45)$$

In the usual development of fluid dynamics, the various terms in expansion Eq. (45) are described as follows. (Google vorticity, Reynolds number, fault site, "Vorticity dynamics") The second term on the RHS of eq. (45) is the effect of expansion of the vorticity field. In an expanding flow, $\underline{\nabla} \cdot \underline{v}$ is positive, and this results in a decrease in the magnitude of vorticity because of the minus sign. The third term is the baroclinic torque. This exists in the atmosphere, which is an incompressible fluid. "This describes viscous diffusion. Turbulence sets in at high Reynolds number, so this term is small in turbulent flows but it is important when the Reynolds number is small. The term $(\underline{w} \cdot \underline{\nabla})\underline{v}$ is the vortex stretching term. This is the most important term in the development of turbulence.

) Eq. (45) can be simplified considerably by writing it as:

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{w} \times \underline{v}) = -\underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} p \right) + \frac{1}{R} \left(\nabla^2 \underline{w} \right) \quad -(46)$$

Now use:

$$\nabla^2 \underline{w} = \underline{\nabla} (\underline{\nabla} \cdot \underline{w}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{w}) \quad -(47)$$

$$= -\underline{\nabla} \times (\underline{\nabla} \times \underline{w})$$

and Kambe's homogeneous field equation:

$$\underline{\nabla} \times \underline{E}_F + \frac{\partial \underline{w}}{\partial t} = \underline{0} \quad -(48)$$

where $\underline{E}_F$ is the fluid electric field of Kambe.

It follows that

$$-\underline{\nabla} \times \underline{E}_F + \underline{\nabla} \times (\underline{w} \times \underline{v}) = -\underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} p \right) - \frac{1}{R} \underline{\nabla} \times (\underline{\nabla} \times \underline{w}) \quad -(49)$$

so

$$\underline{E}_F = \frac{1}{\rho} \underline{\nabla} p + \frac{1}{R} \underline{\nabla} \times \underline{w} + \underline{w} \times \underline{v} \quad -(50)$$

In the Kambe analysis:

10)

$$\underline{E}_F = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla} h = (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(51)$$

where

$$\underline{\nabla} h = \frac{1}{\rho}\underline{\nabla} P \quad -(52)$$

From eqs. (50) to (52):

$$\underline{E}_F = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla} h = \underline{\nabla} h + \frac{1}{R}\underline{\nabla}\times\underline{w} - \underline{w}\times\underline{v} \quad -(53)$$

so

$$\underline{\nabla} h = \frac{1}{\rho}\underline{\nabla} P = \frac{1}{2}\left(-\frac{\partial \underline{v}}{\partial t} - \frac{1}{R}\underline{\nabla}\times\underline{w} + \underline{w}\times\underline{v}\right) \quad -(54)$$

$$= \frac{1}{2}\left((\underline{\nabla}\times\underline{v})\times\underline{v} - \frac{1}{R}\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) - \frac{\partial \underline{v}}{\partial t}\right)$$

the baroclinic term is therefore:

$$\left(\frac{D\underline{w}}{Dt}\right)_{bar} = \frac{1}{\rho}\underline{\nabla}\rho\times\underline{\nabla} h \quad -(55)$$

where

$$\underline{\nabla} h = (\underline{v}\cdot\underline{\nabla})\underline{v} - \frac{\partial \underline{v}}{\partial t} \quad -(56)$$

$$= \underline{E}_F - \partial\underline{v}/\partial t$$

and

$$\underline{\nabla}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\underline{\nabla}\rho \quad -(57)$$