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The wave equation is:

$$\boxed{\Box V^{\mu} = \frac{1}{a_0^2} J^{\mu}} \quad -(1)$$

where

$$V^{\mu} = \left(\frac{\underline{\Phi}}{a_0}, \underline{V} \right) \quad -(2)$$

$$J_F^{\mu} = \left(a_0 q_F, \underline{J}_F \right) -(3)$$

So:

$$\Box \underline{\Phi} = q_F -(4)$$

$$\Box \underline{V} = \frac{1}{a_0^2} \underline{J}_F -(5)$$

Units

$$\underline{\Phi} = m^2 s^{-2}, \quad q_F = s^{-2}, -(6)$$

$$\underline{J}_F = m s^{-3}, \quad a_0 = m s^{-1}, \quad \Box = m^{-2}$$

The continuity equation is:

$$\boxed{\frac{\partial q_F}{\partial t} + \underline{\nabla} \cdot \underline{J}_F = 0} \quad -(7)$$

where

$$q_F = \underline{\nabla} \cdot \left(\left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} \right) -(8)$$

and

$$\underline{J}_F = a_0^2 \underline{\nabla} \times \left(\underline{\nabla} \times \underline{v} \right) - \frac{\partial}{\partial t} \left(\left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} \right) -(9)$$

So

$$\frac{\partial q_F}{\partial t} = \frac{\partial}{\partial t} \left(\underline{\nabla} \cdot \left(\left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} \right) \right) -(10)$$

and

$$\underline{\nabla} \cdot \underline{J}_F = - \underline{\nabla} \cdot \left(\frac{\partial}{\partial t} \left(\underline{\nabla} \cdot \left(\left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} \right) \right) \right) -(11)$$

) because:

$$\underline{\nabla} \cdot \underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = 0, \quad -(12)$$

and
$$\frac{\partial}{\partial t}\left(\underline{\nabla} \cdot ((\underline{v} \cdot \underline{\nabla})\underline{v})\right) = \underline{\nabla} \cdot \left(\frac{\partial}{\partial t} \underline{\nabla} \cdot ((\underline{v} \cdot \underline{\nabla})\underline{v})\right) \quad -(13)$$

Q.E.D.

Define:
$$\partial_\mu = \left(\frac{1}{a_0}\frac{\partial}{\partial t}, \underline{\nabla}\right) \quad -(14)$$

so
$$\boxed{\partial_\mu J^\mu_F = \frac{\partial q_F}{\partial t} + \underline{\nabla} \cdot \underline{J}_F = 0} \quad -(15)$$

Note that eq. (15) is an exact identity.

The Lorenz gauge condition:

$$\partial_\mu v^\mu = \frac{1}{a_0^2}\frac{\partial \Phi}{\partial t} + \underline{\nabla} \cdot \underline{v} = 0 \quad -(16)$$

is an assumption. However it is a very useful assumption because it leads to eqs. (1) to (5). Otherwise these would be much more complicated.

From eqs. (4), (5) and (16):

$$\frac{\partial}{\partial t}(\Box \Phi) + a_0^2 \underline{\nabla} \cdot (\Box \underline{v}) = 0 \quad -(17)$$

By commutativity of differential operators:

$$\Box \left(\frac{\partial \Phi}{\partial t} + a_0^2 \underline{\nabla} \cdot \underline{v}\right) = 0 \quad -(18)$$

The Lorenz gauge (16) is a solution of eq. (18), which is the continuity equation combined with the wave equation, Q. E. D.

This development has an exact parallel in ECE2 electrodynamics, where the Lorenz gauge is known as the radiation gauge.

The great advantage of eq. (1) is that it reduces the whole of fluid dynamics to one second order wave equation.