352(2): Simplification of the Vorticity Equation

The vorticity equation is:

$$\underline{\nabla} \times \underline{w} = R\left(\underline{v} \times \underline{w} - (\underline{v}\cdot\underline{\nabla})\underline{v}\right) \quad -(1)$$

Now use the vector identity (VAPS Problem 8.23):

$$\underline{E} = (\underline{v}\cdot\underline{\nabla})\underline{v} = \frac{1}{2}\underline{\nabla}v^2 - \underline{v}\times(\underline{\nabla}\times\underline{v}) \quad -(2)$$

so:

$$\underline{\nabla}\times\underline{w} = R\left(\underline{v}\times\underline{w} - \frac{1}{2}\underline{\nabla}v^2 + \underline{v}\times\underline{w}\right) \quad -(3)$$

$$\boxed{\underline{\nabla}\times\underline{w} = R\left(2\underline{v}\times\underline{w} - \frac{1}{2}\underline{\nabla}v^2\right)}$$

where

$$\underline{w} = \underline{\nabla}\times\underline{v} \quad -(4)$$

Therefore:

$$\boxed{\begin{aligned}\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) &= \underline{\nabla}(\underline{\nabla}\cdot\underline{v}) - \nabla^2\underline{v} \\ &= R\left(2\underline{v}\times(\underline{\nabla}\times\underline{v}) - \frac{1}{2}\underline{\nabla}v^2\right)\end{aligned}} \quad -(5)$$

For Beltrami flows:

$$\underline{\nabla}\times\underline{v} = R\underline{v} \quad -(6)$$

So

$$\underline{v}\times(\underline{\nabla}\times\underline{v}) = \underline{0} \quad -(7)$$

because $\underline{\nabla}\times\underline{v}$ is parallel to $\underline{v}$. So for Beltrami flows:

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{v}) = \underline{\nabla}(\underline{\nabla}\cdot\underline{v}) - \nabla^2\underline{v}$$

$$= -\frac{R}{2}\underline{\nabla}v^2 \quad -(8)$$

2)

i.e.
$$\nabla^2 \underline{v} - \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) = \frac{R}{2} \underline{\nabla} v^2 \quad - (9)$$

for Beltrami flows.

So Beltrami flows in general are governed by eq. (9).

If the Beltrami flow is incompressible then:
$$\nabla^2 \underline{v} = \frac{R}{2} \underline{\nabla} v^2 \quad - (10)$$

From problem 8-32 of VAPS there exists the following identity:
$$\underline{v} \cdot \underline{\nabla} \times \underline{v} = \underline{v} \cdot \underline{w} = (\underline{v} \times \underline{\nabla}) \cdot \underline{v} \quad - (11)$$

For a uniform vorticity:
$$\underline{w} = \underline{\nabla} \times \underline{v} \quad - (12)$$

then:
$$\underline{v} = \frac{1}{2} \underline{w} \times \underline{r} \quad - (13)$$

in direct analogy to:
$$\underline{B} = \underline{\nabla} \times \underline{A} \quad - (14)$$

and
$$\underline{A} = \frac{1}{2} \underline{B} \times \underline{r} \quad - (15)$$

in the old electrodynamics. From eqs. (11) and (13):
$$\underline{v} \cdot \underline{\nabla} \times \underline{v} = \frac{1}{2} \underline{w} \times \underline{r} \cdot \underline{w} = \frac{1}{2} \underline{w} \times \underline{w} \cdot \underline{r} \quad - (16)$$
$$= \underline{0}$$

Therefore for uniform vorticity:

3) $$\underline{v} \cdot \underline{\nabla} \times \underline{v} = 0 \quad -(17)$$

So for uniform vorticity:

$$\boxed{\underline{v} \cdot \underline{w} = \underline{v} \cdot \underline{\nabla} \times \underline{v} = (\underline{v} \times \underline{\nabla}) \cdot \underline{v} = 0} \quad -(18)$$

i.e. $\underline{w}$ is perpendicular to $\underline{v}$.

In order to convert these equations to fluid electrodynamics, use:

$$\underline{v} = \frac{\rho}{\rho_m} \underline{W} \quad -(19)$$

and $$\underline{w} = \frac{\rho}{\rho_m} \underline{B} \quad -(20)$$

so the vorticity equation of fluid electrodynamics is:

$$\underline{\nabla} \times \left(\frac{\rho}{\rho_m} \underline{B} \right) = R \left(2 \underline{v} \times \left(\frac{\rho}{\rho_m} \underline{B} \right) - \frac{1}{2} \underline{\nabla} \left(\frac{\rho}{\rho_m} \underline{W} \right)^2 \right.$$

$$= R \left(2 \frac{\rho}{\rho_m} \underline{W} \times \left(\frac{\rho}{\rho_m} \underline{B} \right) - \frac{1}{2} \underline{\nabla} \left(\frac{\rho}{\rho_m} \underline{W} \right)^2 \right) \quad -(21)$$

where $$\underline{B} = \underline{\nabla} \times \underline{W} \quad -(22)$$

In general:

$$\underline{\nabla} \times \left(\frac{\rho}{\rho_m} \underline{B} \right) = \frac{\rho}{\rho_m} \underline{\nabla} \times \underline{B} + \underline{\nabla} \left(\frac{\rho}{\rho_m} \right) \times \underline{B} \quad -(22)$$

4) If it is assumed that ρ / ρ_m is independent of distance then:

$$\underline{\nabla}\left(\frac{\rho}{\rho_m}\right) = \underline{0} \quad -(23)$$

In this case eq. (21) simplifies to:

$$\boxed{\underline{\nabla} \times \underline{B} = \frac{\rho}{\rho_m} R \left(2\underline{W} \times (\underline{\nabla} \times \underline{W}) - \frac{1}{2} \underline{\nabla} W^2 \right)} \quad -(24)$$

So if there is a circuit in contact with the vacuum, and if:

$$\underline{W} = \underline{W}(vac) \quad -(25)$$

it is possible that a magnetic flux density $\underline{B}$ appears in the circuit from a turbulent vacuum.
