352(3): Time Dependent Equations

These are: $$\frac{\partial \underline{v}}{\partial t} + \underline{\nabla} h = -(\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(1)$$

$$\frac{\partial h}{\partial t} + a_0^2 \underline{\nabla}\cdot\underline{v} = -\underline{v}\cdot\underline{\nabla} h \quad -(2)$$

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla}\times(\underline{w}\times\underline{v}) = \frac{1}{R}\nabla^2 \underline{w} \quad -(3)$$

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla}\times\underline{E} = \underline{0} \quad -(4)$$

$$\underline{E} = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla} h \quad -(5)$$

$$= (\underline{v}\cdot\underline{\nabla})\underline{v}$$

$$a_0^2 \underline{\nabla}\times\underline{H} - \frac{\partial \underline{E}}{\partial t} = \underline{J} \quad -(6)$$

$$\underline{H} = \underline{w} \quad -(7)$$

where

and $$\underline{J} = \frac{\partial^2 \underline{v}}{\partial t^2} + \underline{\nabla}\frac{\partial h}{\partial t} + a_0^2 \underline{\nabla}\times(\underline{\nabla}\times\underline{v}), \quad -(8)$$

$$\underline{\nabla} h = \frac{1}{\rho}\Delta \underline{p} \quad -(9)$$

From eqs. (4) and (5):

$$\frac{\partial \underline{w}}{\partial t} - \frac{\partial \underline{\nabla}\times\underline{v}}{\partial t} - \underline{\nabla}\times\underline{\nabla} h = 0 \quad -(10)$$

i.e $$\frac{\partial \underline{w}}{\partial t} = \underline{\nabla}\times\frac{\partial \underline{v}}{\partial t} \quad -(11)$$

which is consistent with:

$$\underline{w} = \underline{\nabla}\times\underline{v}. \quad -(12)$$

Also from eqs. (4) and (5):

2)

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{v} \cdot \underline{\nabla}) \underline{v} = \underline{0} \quad -(13)$$

From eqs. (12) and (13):

$$\boxed{\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = \underline{0}} \quad -(14)$$

is a particular solution. Eq. (14) means that:

$$\underline{\nabla} h = \frac{1}{\rho} \underline{\Delta p} = \underline{0} \quad -(15)$$

and there is no change of pressure.

Also from eq. (4):

$$\underline{\nabla} \times \frac{\partial \underline{v}}{\partial t} + \underline{\nabla} \times \underline{E} = \underline{0}, \quad -(16)$$

a particular solution of which is:

$$\frac{\partial \underline{v}}{\partial t} + \underline{E} = \underline{0}. \quad -(17)$$

If

$$\underline{E} = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla} h, \quad -(18)$$

eqs. (17) and (18) imply:

$$\underline{\nabla} h = 0 \quad -(19)$$

again. The simple flow of Eq. (14) can be graphed and animated as a preliminary exercise.

The Beltrami solution of Eq. (3) is

$$\boxed{\frac{\partial \underline{w}}{\partial t} = \frac{1}{R} \nabla^2 \underline{w}} \quad -(20)$$

because for Beltrami flow:

$$\underline{W} \times \underline{V} = \underline{0} \quad -(21)$$

and

$$\underline{\nabla} \times \underline{V} = k\underline{V} \quad -(22)$$

The simple equation (20) can also be graphed and animated. At a certain Reynolds number the vorticity of Beltrami flow develops turbulence.

In general, eq. (3) is:

$$\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{V}) + \underline{\nabla} \times (\underline{W} \times \underline{V}) = \frac{1}{R} \nabla^2 \underline{W} \quad -(23)$$

$$= \frac{1}{R}\left(\underline{\nabla}(\underline{\nabla} \cdot \underline{W}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{W})\right)$$

in which

$$\underline{\nabla} \cdot \underline{W} = \underline{\nabla} \cdot \underline{\nabla} \times \underline{V} = 0 \quad -(24)$$

Therefore a particular solution of eq. (23) is:

$$\boxed{\frac{\partial \underline{V}}{\partial t} + \underline{W} \times \underline{V} = -\frac{1}{R} \underline{\nabla} \times \underline{W}} \quad -(25)$$

so

$$\underline{\nabla} \times \underline{W} = -R\left(\frac{\partial \underline{V}}{\partial t} + \underline{W} \times \underline{V}\right) \quad -(26)$$

For Beltrami flow:

$$\underline{\nabla} \times \underline{W} = -R \frac{\partial \underline{V}}{\partial t} \quad -(27)$$

4) Therefore Beltrami flow is described by eqs (20) and (27):

$$\nabla^2 \underline{w} = R \frac{\partial \underline{w}}{\partial t} \quad -(28)$$

and

$$\underline{\nabla} \times \underline{w} = -R \frac{\partial \underline{v}}{\partial t} \quad -(29)$$

with

$$\underline{w} = \underline{\nabla} \times \underline{v} \quad -(30)$$

Therefore

$$\nabla^2 \underline{v} = R \frac{\partial \underline{v}}{\partial t} \quad -(31)$$

and

$$\underline{\nabla} \times \underline{w} = -R \frac{\partial \underline{v}}{\partial t} \quad -(32)$$

$$= \underline{\nabla} \times (\underline{\nabla} \times \underline{v})$$

$$= \underline{\nabla} (\underline{\nabla} \cdot \underline{v}) - \nabla^2 \underline{v} \quad -(33)$$

These equations imply that Beltrami flow is incompressible and a inviscid because:

$$\underline{\nabla} \cdot \underline{v} = 0 \quad -(34)$$

The interesting features are obtained therefore when:

$$\underline{w} \times \underline{v} \neq \underline{0} \quad -(35)$$

and

$$\frac{\partial \underline{v}}{\partial t} + (\underline{\nabla} \times \underline{v}) \times \underline{v} = -\frac{1}{R} \underline{\nabla} \times (\underline{\nabla} \times \underline{v}) \quad -(36)$$