5.2(4): Simplification of the Vorticity Equation

Consider the Kambe equations:

$$\frac{\partial \underline{v}}{\partial t} + \underline{\nabla} h = -(\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(1)$$

$$\frac{\partial h}{\partial t} + a_0^2\,\underline{\nabla}\cdot\underline{v} = -\underline{v}\cdot\underline{\nabla} h \quad -(2)$$

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla}\times(\underline{w}\times\underline{v}) = \underline{0} \quad -(3)$$

Eqs (1) and (2) can be written as:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\underline{\nabla} h \quad -(4)$$

and

$$\frac{Dh}{Dt} = \frac{\partial h}{\partial t} + (\underline{v}\cdot\underline{\nabla})h = -a^2\,\underline{\nabla}\cdot\underline{v} \quad -(5)$$

The correct version of eq. (3) is:

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla}\times(\underline{w}\times\underline{v}) = \frac{1}{R}\nabla^2\underline{w}. \quad -(6)$$

Now apply the vector identity:

$$\underline{\nabla}\times(\underline{w}\times\underline{v}) = \underline{w}(\underline{\nabla}\cdot\underline{v}) - (\underline{\nabla}\cdot\underline{w})\underline{v} + (\underline{v}\cdot\underline{\nabla})\underline{w} - (\underline{w}\cdot\underline{\nabla})\underline{v} \quad -(7)$$

in which: $\underline{\nabla}\cdot\underline{w} = \underline{\nabla}\cdot\underline{\nabla}\times\underline{v} = 0 \quad -(8)$

and $\underline{w}\cdot\underline{\nabla} = \underline{\nabla}\times\underline{v}\cdot\underline{\nabla} \neq 0 \quad -(9)$

It follows that the vorticity equation (6) simplifies to:

2)

$$\boxed{\frac{D\underline{w}}{Dt} = \frac{1}{R}\nabla^2\underline{w} - \underline{w}\left(\underline{\nabla}\cdot\underline{v}\right) - \left(\underline{w}\cdot\underline{\nabla}\right)\underline{v}} \quad - (10)$$

in general, in which:

$$\frac{D\underline{w}}{Dt} = \frac{\partial \underline{w}}{\partial t} + \left(\underline{v}\cdot\underline{\nabla}\right)\underline{w} \quad - (11)$$

is the convective derivative of vorticity.

For an incompressible, inviscid fluid:

$$\underline{\nabla}\cdot\underline{v} = 0 \quad - (12)$$

so

$$R\frac{D\underline{w}}{Dt} = \nabla^2\underline{w} - R\left(\underline{w}\cdot\underline{\nabla}\right)\underline{v} \quad - (13)$$

Kambe's version of Eq. (10) is:

$$\frac{D\underline{w}}{Dt} = -\underline{w}\left(\underline{\nabla}\cdot\underline{v}\right) - \left(\underline{w}\cdot\underline{\nabla}\right)\underline{v} \quad - (14)$$