341(2): Vorticity of the Newtonian Orbit

The gravitomagnetic field $\underline{\Omega}_g$ is a vorticity defined by:

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_g \quad -(1)$$

where $\underline{v}_g$ is a linear velocity. In order to calculate the vorticity of an orbit assume that $\underline{v}_g$ is the orbital linear velocity of the orbit. If it is assumed that $\underline{\Omega}_g$ is uniform then:

$$\underline{v}_g = \frac{1}{2}\underline{\Omega}_g \times \underline{r} \quad -(2)$$

as in previous notes. It follows that:

$$v_g^2 = \frac{1}{4}\left(\underline{\Omega}_g \times \underline{r} \cdot \underline{\Omega}_g \times \underline{r}\right) \quad -(3)$$

$$= \frac{1}{4}\left(\Omega_g^2 r^2 - (\underline{\Omega}_r \cdot \underline{r})(\underline{r} \cdot \underline{\Omega}_g)\right)$$

If

$$\underline{\Omega}_r \perp \underline{r} \quad -(4)$$

then:

$$v_g^2 = \frac{1}{4}\Omega_g^2 r^2 \quad -(5)$$

so

$$\Omega_g^2 = 4\frac{v_g^2}{r^2}. \quad -(6)$$

For the Newtonian orbit:

$$\Omega_g^2 = \frac{4MG}{r^2}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(7)$$

where a is the semi major axis of the ellipse.

The Newtonian orbit is described by:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad -(8)$$

2) where α is the half right latitude and ϵ the eccentricity. In plane polar coordinates the linear velocity is described by:

$$\underline{v}_g = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta, \quad -(9)$$

and if T is the time taken for one revolution of the orbit:

$$\Omega_g T = 2\pi \quad -(10)$$

Therefore the vorticity of the orbit is:

$$\Omega_g^2 = \frac{4}{r^2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) \quad -(11)$$

$$= \frac{4v_g^2}{r^2}$$

From a Lagrangian analysis:

$$L = mr^2\frac{d\theta}{dt} \quad -(12)$$

and

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(13)$$

It follows that:

$$\Omega_g^2 = \frac{4}{r^2}\left(\frac{d\theta}{dt}\right)^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(14)$$

$$= \frac{4L^2}{m^2r^6}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)$$

The effect of a precession is to increment the vorticity as follows:

$$\Omega_g^2 \rightarrow \frac{4}{r^2}\left(v_g^2 + v_g^2(\text{precession})\right) \quad -(15)$$

So:

$$v_{g1}^2 = \frac{L^2}{m^2 r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) + \Omega^2 r^2 \quad -(16)$$

where Ω is the observed precession:

$$\Omega = \frac{\Omega_g}{2} \quad -(17)$$

Now define the orbit with precession as:

$$v_{g1}^2 = \frac{L^2}{m^2 r^4}\left(\left(\frac{dr}{d\theta}\right)_1^2 + r^2\right) \quad -(18)$$

It follows from eqs. (16) to (18) that:

$$\left(\frac{dr}{d\theta}\right)_1^2 = \left(\frac{dr}{d\theta}\right)^2 + \frac{m^2\Omega^2 r^6}{L^2} \quad -(19)$$

Using eq. (8):

$$\frac{dr}{d\theta} = \frac{\epsilon}{\alpha} r^2 \sin\theta \quad -(20)$$

so

$$\left(\frac{dr}{d\theta}\right)_1^2 = \left(\frac{\epsilon^2}{\alpha^2}\sin^2\theta\right) r^4 + \left(\frac{m\Omega r^3}{L}\right)^2 \quad -(21)$$

Here:

$$\sin^2\theta = 1 - \cos^2\theta \quad -(22)$$

$$= 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2$$

From these equations it is possible to find the

4) equation:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{\epsilon^2}{\alpha^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2\right)\right)+\left(\frac{m\Omega r^3}{L}\right)^2 \quad -(23)$$

so:

$$d\theta = \left(r^4\left(\frac{\epsilon^2}{\alpha^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2\right)\right)+\left(\frac{m\Omega r^3}{L}\right)^2\right)^{-1/2} dr \quad -(24)$$

It is possible in theory to numerically integrate eq. (24) to give θ as a function of r for the new orbit.