347(5): Orbital Precession as a Lorentz Force Equation

Consider the minimal prescription in electrodynamics:

$$\underline{p} \rightarrow \underline{p} + e\underline{A} \quad -(1)$$

where p is the momentum and $\underline{A}$ the vector potential. Here e is the modulus of the charge on the electron. As in Note 347(2) the gravitomagnetic minimal prescription is:

$$\underline{p} \rightarrow \underline{p} + m\underline{v}_g \quad -(2)$$

where the gravitomagnetic vector potential is:

$$\underline{W}_g = \underline{v}_g \quad -(3)$$

The free particle hamiltonian is:

$$H = \frac{1}{2m}\left(\underline{p} + m\underline{v}_g\right)\cdot\left(\underline{p} + m\underline{v}_g\right) \quad -(4)$$

$$= \frac{p^2}{2m} + \frac{1}{2}mv_g^2 + \frac{1}{2}\underline{L}\cdot\underline{\Omega}_g$$

where

$$\underline{L} = \underline{p} \times \underline{r} \quad -(5)$$

and

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_r \quad -(6)$$

is the gravitomagnetic field. The orbital precession frequency is the Larmor precession frequency:

$$\Omega = \frac{1}{2}\Omega_g \quad -(7)$$

In general Ω is any experimentally measured precession frequency.

In the presence of a gravitational potential:

$$U(r) = -\frac{mMG}{r} \quad -(8)$$

2) the Hamiltonian becomes:

$$H = \frac{1}{2m}\left(\underline{p} + m\underline{v}_g\right)\cdot\left(\underline{p} + m\underline{v}_g\right) + U(r) \quad -(9)$$

In the absence of a gravitomagnetic field the Hamiltonian reduces to:

$$H = \frac{1}{2m}\,\underline{p}\cdot\underline{p} + U(r) \quad -(10)$$

It is well known that the Hamiltonian (10) produces the conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(11)$$

in plane polar coordinates (r, θ), where α is the half right latitude and ϵ the eccentricity.

Therefore the gravitomagnetic field (6) produces the orbital precession frequency Ω, with Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}mv_g^2 + \frac{1}{2}\underline{L}\cdot\underline{\Omega}_g + U(r) \quad -(12)$$

The Lagrangian is calculated from the Hamiltonian using the canonical momentum:

$$p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} \quad -(13)$$

Denote:

$$\underline{\dot{r}} = \frac{1}{m}\left(\underline{p} + m\underline{v}_g\right) \quad -(14)$$

then:

$$H = \underline{p}\cdot\underline{\dot{r}} - \mathcal{L} \quad -(15)$$

The Lagrangian is therefore:

3)
$$\mathcal{L} = \frac{1}{2}m\left(\underline{p}+m\underline{v}_g\right)\cdot\left(\underline{p}+m\underline{v}_g\right) - U(r) - m\underline{\dot{r}}\cdot\underline{v}_g \quad -(16)$$

and the relevant Euler Lagrange equation is:

$$\nabla\mathcal{L} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\underline{\dot{r}}}\right) \quad -(17)$$

The left hand side of this equation is:

$$\underline{\nabla}\mathcal{L} = \underline{\nabla}\left(\frac{1}{2}m\underline{\dot{r}}\cdot\underline{\dot{r}} - U(r)\right) - m\underline{\nabla}\left(\underline{\dot{r}}\cdot\underline{v}_g\right) \quad -(18)$$

$$= -\underline{\nabla}U(r) - m\underline{\nabla}\left(\underline{\dot{r}}\cdot\underline{v}_g\right)$$

In general:

$$\underline{\nabla}\left(\underline{\dot{r}}\cdot\underline{v}_g\right) = \left(\underline{\dot{r}}\cdot\underline{\nabla}\right)\underline{v}_g + \left(\underline{v}_g\cdot\underline{\nabla}\right)\underline{\dot{r}} + \underline{\dot{r}}\times\left(\underline{\nabla}\times\underline{v}_g\right) + \underline{v}_g\times\left(\underline{\nabla}\times\underline{\dot{r}}\right) \quad -(19)$$

$$= \left(\underline{\dot{r}}\cdot\underline{\nabla}\right)\underline{v}_g + \underline{\dot{r}}\times\left(\underline{\nabla}\times\underline{v}_g\right)$$

if it is assumed that:

$$\left(\underline{v}_g\cdot\underline{\nabla}\right)\underline{\dot{r}} = \underline{0} \quad -(20)$$

and

$$\underline{\nabla}\times\underline{\dot{r}} = \underline{0} \quad -(21)$$

So:

$$\underline{\nabla}\mathcal{L} = -\underline{\nabla}U(r) - m\left(\left(\underline{\dot{r}}\cdot\underline{\nabla}\right)\underline{v}_g + \underline{\dot{r}}\times\left(\underline{\nabla}\times\underline{v}_g\right)\right) \quad -(22)$$

The right hand side of eq. (17) is:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}}\right) = \frac{d}{dt}\left(m\dot{\underline{r}} - m\underline{v}_g\right) \quad -(23)$$

$$= m\ddot{\underline{r}} - m\frac{d\underline{v}_g}{dt}$$

In order to evaluate the total derivative of $\underline{v}_g$, consider one component, so for example:

$$\frac{dv_{gx}}{dt} = \frac{\partial v_{gx}}{\partial t} + \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial v_{gx}}{\partial x}\right) + \cdots \quad -(24)$$

to first order.

Therefore in three dimensions:

$$\frac{d\underline{v}_g}{dt} = \frac{\partial \underline{v}_g}{\partial t} + (\dot{\underline{r}}\cdot\underline{\nabla})\underline{v}_g \quad -(25)$$

Therefore:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}}\right) = m\ddot{\underline{r}} - m\left(\frac{\partial \underline{v}_g}{\partial t} + (\dot{\underline{r}}\cdot\underline{\nabla})\underline{v}_g\right) \quad -(26)$$

The Euler Lagrange equation is therefore:

$$-\underline{\nabla}U(r) - m\dot{\underline{r}}\times(\underline{\nabla}\times\underline{v}_g) = m\ddot{\underline{r}} - m\frac{\partial \underline{v}_g}{\partial t} \quad -(27)$$

i.e.

$$m\ddot{\underline{r}} = -\underline{\nabla}U(r) + m\frac{\partial \underline{v}_g}{\partial t} - m\dot{\underline{r}}\times(\underline{\nabla}\times\underline{v}_g) \quad -(28)$$

5) Now write

$$m\phi_g = -U(r), \quad -(29)$$

and

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_g \quad -(30)$$

to find the gravitomagnetic Lorentz force law

$$\boxed{\underline{F} = m\ddot{\underline{r}} = -m\left(\underline{E}_g + \dot{\underline{r}} \times \underline{\Omega}_g\right)} \quad -(31)$$

where

$$\underline{E}_g = -\underline{\nabla}\phi_g - \frac{\partial \underline{v}_g}{\partial t} \quad -(32)$$

The precession of an orbit is therefore governed by the Lorentz force law (31) with precession frequency

$$\underline{\Omega} = \frac{1}{2}\underline{\Omega}_g \quad -(33)$$

The Hamiltonian corresponding to eq. (31) is:

$$H = \frac{1}{2m}\left(\underline{p} + m\underline{v}_g\right)\cdot\left(\underline{p} + m\underline{v}_g\right) + U(r) \quad -(34)$$

with the gravitational potential energy:

$$U(r) = -\frac{mMG}{r} \quad -(35)$$

so from eqs. (29) and (35):

$$\phi_g := \frac{MG}{r} \quad -(36)$$

It follows that

$$\underline{\nabla}\phi_g = -\frac{MG}{r^2}\underline{e}_r \quad -(37)$$

6) So
$$\underline{E}_g = \frac{MG}{r^2}\underline{e}_r - \frac{\partial \underline{v}_g}{\partial t} \quad -(38)$$

and
$$\boxed{\underline{F} = m\ddot{\underline{r}} - \frac{mMG}{r^2}\underline{e}_r + m\frac{\partial \underline{v}_g}{\partial t} - m\dot{\underline{r}} \times \underline{\Omega}_g \quad -(39)}$$

where
$$\dot{\underline{r}} = \frac{1}{m}\left(\underline{p} + m\underline{v}_g\right) \quad -(40)$$

In the absence of the gravitomagnetic field, eq. (39) reduces to:
$$\underline{F} = m\ddot{\underline{r}} - \frac{mMG}{r^2}\underline{e}_r \quad -(41)$$

In these equations for a planar orbit:
$$\underline{v} = \dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\underline{e}_r\right) \quad -(42)$$
$$= \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

~~in the absence of a gravitomagnetic field.~~ The acceleration is the absence of a gravitomagnetic field is:
$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta\right) = \ddot{\underline{r}} \quad -(43)$$

Eq. (43) gives rise to the well known Coriolis and centripetal forces, as in previous UFT papers. So the gravitomagnetic force occurs in addition to these forces.

7) The gravitomagnetic field is governed by the law:

$$\underline{\nabla} \times \underline{\Omega}_g = \frac{4\pi G}{c^2} \underline{J}_g \quad -(44)$$

analogous to the Ampère Law:

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(45)$$

in electrodynamics and magnetostatics. The vacuum gravitomagnetic permeability is:

$$\mu_{g0} = \frac{4\pi G}{c^2} \quad -(46)$$

The gravitomagnetic four potential is:

$$W^\mu = (\phi_g, c\underline{W}_g) \quad -(47)$$

where

$$\underline{W}_g = \underline{v}_g \quad -(48)$$