347(4): Development of the Lagrangian and Hamiltonian.

The gravitomagnetic field is defined by the vorticity:

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_g \quad -(1)$$

where in plane polar coordinates:

$$\underline{v}_g = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(2)$$

$$= \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}$$

Here

$$\underline{\omega} \times \underline{r} = r\dot{\theta}\,\underline{e}_\theta. \quad -(3)$$

Therefore:

$$\underline{\Omega}_g = \underline{\nabla} \times \frac{d\underline{r}}{dt} + \underline{\nabla} \times (\underline{\omega} \times \underline{r}) \quad -(4)$$

$$= \frac{d}{dt}(\underline{\nabla} \times \underline{r}) + \underline{\nabla} \times (r\dot{\theta}\,\underline{e}_\theta),$$

where we have used:

$$\underline{\nabla} \times \frac{d\underline{r}}{dt} = \frac{d}{dt}(\underline{\nabla} \times \underline{r}). \quad -(5)$$

If a function is expressed in cylindrical polar coordinates:

$$\underline{F} = F_r\,\underline{e}_r + F_\theta\,\underline{e}_\theta + F_z\,\underline{k} \quad -(6)$$

then:

$$\underline{\nabla} \times \underline{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial Z}\right)\underline{e}_r + \left(\frac{\partial F_r}{\partial Z} - \frac{\partial F_z}{\partial r}\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\underline{e}_z \quad -(7)$$

Therefore:

$$\underline{\nabla} \times \underline{r} = 0 \quad -(8)$$

$$= \frac{\partial r}{\partial Z}\,\underline{e}_\theta - \frac{1}{r}\frac{\partial r}{\partial \theta}\,\underline{k}$$

) For planar orbits:

$$\partial r / \partial Z = 0 \quad -(9)$$

So

$$\underline{\nabla} \times \underline{r} = -\frac{1}{r}\frac{\partial r}{\partial \theta}\,\underline{k} \quad -(10)$$

Similarly:

$$\underline{\nabla} \times (r\dot{\theta}\,\underline{e}_\theta) = \underline{\nabla} \times F_\theta \underline{e}_\theta$$

$$= -\frac{\partial F_\theta}{\partial Z}\underline{e}_r + \frac{1}{r}\left(\frac{\partial (r F_\theta)}{\partial r}\right)\underline{e}_Z \quad -(11)$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}(r^2\dot{\theta})\right)\underline{k}$$

It follows that:

$$\underline{\Omega_g} = \left(-\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial \theta}\right) + \frac{1}{r}\left(\frac{\partial}{\partial r}(r^2\dot{\theta})\right)\right)\underline{k} \quad -(12)$$

$$= \left(-\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial \theta}\right) + 2\dot{\theta} + r\frac{\partial \dot{\theta}}{\partial r}\right)\underline{k}$$

The precession is therefore:

$$\boxed{\underline{\Omega} = \left(\dot{\theta} + \frac{r}{2}\frac{\partial \dot{\theta}}{\partial r} - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial \theta}\right)\right) \quad -(13)}$$

In general:

$$\underline{\Omega}_g = \frac{d}{dt}\underline{\nabla} \times \underline{r} + \underline{\nabla} \times (\underline{\omega} \times \underline{r}) \quad -(14)$$

so precession is due to added vorticity.

3) The Hamiltonian is:

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}mv_g^2 + \Omega L_0 + U \quad -(15)$$

$$:= \frac{1}{2}m(v_0^2 + v_g^2) + mv_0v_g + U$$

Self consistently:

$$\Omega L_0 = mv_0v_g = mv_0\Omega r \quad -(16)$$

So:

$$L_0 = mv_0 r \quad -(17)$$

In the first approximation:

$$H = \frac{1}{2}mv^2 + U(r) \quad -(18)$$

where

$$v^2 = v_0^2 + v_g^2 \quad -(19)$$

Eq. (18) gives the conic section without precession:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(20)$$

where

$$\alpha = \frac{L^2}{m^2 mG}, \quad -(21)$$

$$\epsilon = \left(1 + \frac{2EL^2}{m^3 m^2 G^2}\right)^{1/2} \quad -(22)$$

The Lagrangian corresponding to eq. (18) is:

$$\mathcal{L} = \frac{1}{2}mv^2 - U(r) \quad -(23)$$

which gives

$$L = mr^2\dot{\theta} = \text{constant} \quad -(24)$$

i.e.

$$\frac{dL}{dt} = 0 \quad -(25)$$

4) Therefore the hamiltonian (18) can be expressed as:

$$H = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + \bar{U}(r) \quad -(26)$$

The precession term changes this to:

$$H = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + \Omega L_0 + \bar{U}(r) \quad -(27)$$

where

$$\Omega = \frac{v_g}{r} \quad -(28)$$

Here L_0 is the constant of motion defined by the unperturbed hamiltonian:

$$H_0 = \frac{1}{2} m v_0^2 + U(r) \quad -(29)$$

Finally assume that L_0 is constant, so:

$$H = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + \frac{v_g L_0}{r} + \bar{U}(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + \frac{1}{r} \left(v_g L_0 - mMG \right) \quad -(30)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + \frac{1}{r} \left(\Omega r L_0 - mMG \right)$$

In general:

$$H = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + L_0 \Omega(r) + U(r) \quad -(31)$$

and

$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + L_0 \Omega(r) - \bar{U}(r) \quad -(32)$$