345(7): Calculation of the Lense-Thirring Effect in a Polar Orbit.

From note 345(5) the Lense Thirring precession from ECE2 theory is calculated from the gravitomagnetic field:

$$\underline{\Omega} = \frac{2}{5} \frac{MGR^2}{c^2 r^3} \left(\underline{\omega} - 3\underline{n} \left(\underline{\omega} \cdot \underline{n} \right) \right) \quad -(1)$$

where

$$\underline{n} = \frac{\underline{r}}{r} \quad -(2)$$

in the notation of note 345(5). In that note it was assumed that

$$\underline{\omega} \cdot \underline{n} = 0 \quad -(2)$$

for the sake of simplicity. This produced a Lense-Thirring precession of

$$\Omega_{LT} = 49.4 \text{ milliarcseconds a year} \quad -(3)$$

compared with the experimental claim of

$$\Omega(\mathrm{exp}) = (40.9 \pm 7.8) \text{ milliarcseconds per year.} \quad -(4)$$

A more accurate theory has to account for the polar orbit of Gravity Probe B, which therefore orbits in the plane:

$$\underline{r} = Y\underline{j} + Z\underline{k}. \quad -(4)$$

At the north pole $\underline{r} = Z\underline{k}$. At the south pole $\underline{r} = -Z\underline{k}$, at the equator, $\underline{r} = \pm Y\underline{j}$. The orbit was a geocentric polar orbit every ninety minutes, at 90° to the plane of the equator.

The axis of spin of the earth is:

$$\underline{\omega} = \omega \underline{k} \quad -(5)$$

Therefore:
$$\underline{\omega} \cdot \underline{n} = \frac{Z}{r} \omega \underline{k} \quad -(6)$$

and
$$3\underline{n}(\underline{\omega} \cdot \underline{n}) = \frac{3\underline{r} Z \omega}{r^2}$$
$$= 3(Y\underline{j} + Z\underline{k})\frac{Z\omega}{r^2} \quad -(7)$$

It follows that:
$$\underline{\omega} - 3\underline{n}(\underline{\omega} \cdot \underline{n})$$
$$= \omega \underline{k} - 3\frac{\omega Z}{r}\left(\frac{Y\underline{j}}{r} + \frac{Z\underline{k}}{r}\right)$$
$$= \omega\left(\left(1 - \frac{3Z^2}{r^2}\right)\underline{k} - \frac{3YZ}{r^2}\underline{j}\right) \quad -(8)$$

Define:
$$\sin\theta = \frac{Z}{r}, \quad \cos\theta = \frac{Y}{r} \quad -(9)$$

Then:
$$\underline{\omega} - 3\underline{n}(\underline{\omega} \cdot \underline{n}) = \omega\left(\left(1 - 3\sin^2\theta\right)\underline{k} - 3\sin\theta\cos\theta\,\underline{j}\right) \quad -(10)$$

At the equator:
$$\sin\theta = 0 \quad -(11)$$

so
$$\underline{\omega} = \omega \underline{k} \quad -(12)$$

At the north pole:
$$\sin\theta = 1 \quad -(13)$$

and
$$\cos\theta = 0 \quad -(14)$$

so
$$\underline{\omega} = -2\omega \underline{k} \quad -(15)$$

3) The result claimed by NASA / Stanford is:

$$\Omega_{LT} = 0.0409 \text{ arc seconds a year} = 6.284 \times 10^{-15} \text{ radians per second} \quad -(16)$$

The result for

$$\underline{\omega} = \omega \underline{k} \quad -(17)$$

is 49.4 milliarcseconds a year or

$$\Omega_{LT} = 7.59 \times 10^{-15} \text{ rad } s^{-1} \quad -(18)$$

So:

$$\left| \underline{\omega} - 3\underline{n}\left(\underline{\omega} \cdot \underline{n}\right) \right| = \frac{6.284}{7.59}\, \omega \quad -(19)$$

in order that the experimental result be reproduced exactly, i.e.

$$\left| \underline{\omega} - 3\underline{n}\left(\underline{\omega} \cdot \underline{n}\right) \right| = \omega \left(\left(1 - 3\sin^2\theta\right)^2 + 3\sin^2\theta\cos^2\theta \right)^{1/2}$$

$$= 0.8279\,\omega \quad -(20)$$

So

$$\left(1 - 3\sin^2\theta\right)^2 + 3\sin^2\theta\cos^2\theta = 0.6854 \quad -(21)$$

i.e.

$$\left(1 - 3\sin^2\theta\right)^2 + 3\sin^2\theta\left(1 - \sin^2\theta\right) = 0.6854$$

$$1 - 6\sin^2\theta + 9\sin^4\theta + 3\sin^2\theta - 3\sin^4\theta = 0.6854$$

$$6\sin^4\theta - 3\sin^2\theta + 0.3146 = 0 \quad -(22)$$

Denote

$$x = \sin^2\theta \quad -(23)$$

then

$$x^2 - 0.5x + 0.0524 = 0 \quad -(24)$$

4) So:

$$x = \frac{1}{2}\left(0.5 \pm \left(0.25 - 0.21\right)^{1/2}\right) \quad -(25)$$

$$= \frac{1}{2}\left(0.5 \pm 0.2\right)$$

So

$$\sin^2\theta = 0.35 \quad -(26)$$

or

$$\sin^2\theta = 0.15 \quad -(27)$$

i.e

$$\sin\theta = 0.5916 \quad -(28)$$

or

$$\sin\theta = 0.3873 \quad -(29)$$

Therefore at

$$\frac{Z}{r} = 0.5916 \text{ or } 0.3873 \quad -(30)$$

the exact experimental result is obtained. In other words it is obtained at two latitudes defined by eq. (30). It is by no means clear how the experimental Lense Thirring effect is worked out and separated from the geodetic effect.

So within the dipole approximation, this analysis reproduces the experimental claim exactly.