345(A): Calculation of the Earth's Geodetic Precession with ECEF

The calculation proceeds in close analogy with that of the Lense Thirring effect, which gave:

$$\Omega = 36.3 \text{ milliarcseconds a year} \quad -(1)$$

in comparison with the experimental result from Gravity Probe B of:

$$\Omega(exp) = (40.9 \pm 7.8) \text{ milliarcseconds per year} \quad -(2)$$

Consider the earth to be a static sphere in a rotating frame of reference. Its gravitomagnetic field in the dipole approximation is:

$$\underline{\Omega} = \frac{2G}{c^2 r^3}\left(\underline{L} - 3\left(\underline{L}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right) \quad -(3)$$

in which the angular momentum $\underline{L}$ is that of Gravity Probe B:

$$L = MrV \quad -(4)$$

where M is the mass of the earth, r is the distance from gravity Probe B to the centre of the earth:

$$r = 7.02 \times 10^6 \text{ metres} \quad -(5)$$

and where V is the orbital linear velocity of gravity Probe B:

$$V = \frac{2\pi r}{T} \quad -(6)$$

where T is its period of rotation.

Assuming for simplicity that:

$$\underline{L} \cdot \underline{r} = 0 \quad -(7)$$

then:

$$\underline{\Omega} = \frac{2G}{c^2 r^3} \underline{L}$$

$$= \frac{2MG}{c^2 r^2} \underline{v} \quad -(8)$$

So the relevant gravitomagnetic field magnitude is

$$\Omega = \frac{2gv}{c^2} \quad -(9)$$

where the acceleration due to gravity is:

$$g = \frac{MG}{r^2} \quad -(10)$$

$$= 9.8066 \text{ m s}^{-2}$$

The Larmor precession frequency due to the torque between $\underline{\Omega}$ and the gravitomagnetic dipole moment (gyroscope) or Lord Kelvin's Probe B is

$$\Omega_L = \frac{1}{2} \Omega \quad -(11)$$

assuming a gravitomagnetic Landé factor of unity

So the geodetic precession is:

$$\boxed{\Omega_L = \frac{gv}{c^2}} \quad -(12)$$

Therefore:

$$\Omega_L = 1.0911 \times 10^{-16} \text{ V rad s}^{-1} \quad -(13)$$

$$= 3.443 \times 10^{-9} \text{ V rad per year}$$

$$= 7.10 \times 10^{-4} \text{ V arc seconds per year}$$

The time T taken for one orbit of Gravity Probe B is 90 minutes = 5,400 seconds. So

$$v = \frac{2\pi \times 7.02 \times 10^{6} \text{ m s}^{-1}}{5.4 \times 10^{3}} \quad -(14)$$

$$= 8168 \text{ m s}^{-1}$$

So $\boxed{\Omega_L = 5.80 \text{ arc seconds a year}} \quad -(15)$

The experimental result from Gravity Probe B is $\Omega(\text{exp}) = 6.6144$ arc seconds a year $-(16)$

This is satisfactory agreement, bearing in mind the simplifying assumptions.

So ECE2 gives both the geodesic effect and the Lense-Thirring effect from the gravitomagnetic field.