345(5): Final Version of the Calculation of the Lense - Thirring Effect.

This is based on the calculation of UFT 117, with reference to:

H. Pfister, http://philsci-archive.pitt.edu/archive/00002681/01/Lense from the Institute of Theoretical Physics, University of Tuebingen. The appendix and online notes for UFT 117 give all the details.

Consider the Earth's gravitomagnetic field $\underline{\Omega}$ in the dipole approximation:

$$\underline{\Omega} = \frac{2}{5} \frac{MGR^2}{c^2 r^3} \left(\underline{\omega} - 3\underline{n} \left(\underline{\omega} \cdot \underline{n} \right) \right) \quad -(1)$$

where

$$\underline{n} = \frac{\underline{r}}{r} \quad -(2)$$

and $\underline{\omega}$ is the angular velocity vector of the Earth. Here M is the mass of the earth, G is Newton's constant, R is the radius of the earth, r is the distance from the centre of the earth to the Gravity Probe B satellite, which is in polar orbit. So to a good approximation:

$$\underline{\omega} \cdot \underline{n} = 0 \quad -(3)$$

so

$$\underline{\Omega} = \frac{2}{5} \frac{MGR^2}{c^2 r^3} \underline{\omega} \quad -(4)$$

2) The Gravity Probe B spacecraft carries a gyroscope, with gravitomagnetic dipole moment $\underline{m}$. The torque between $\underline{m}$ and $\underline{\Omega}$ is:

$$\underline{Tq} = \underline{m} \times \underline{\Omega} \quad -(5)$$

giving a Larmor precession frequency:

$$\boxed{\Omega_{LT} = \frac{1}{2}\Omega} \quad -(6)$$

The Lense Thirring precession is Ω_{LT} in radians per second. We have:

$$\Omega = \frac{2}{5}\frac{MG}{c^2 r^3} R^2 \omega \quad -(7)$$

where

$$R = 6.37 \times 10^6 \, m$$
$$r = 7.02 \times 10^6 \, m$$
$$c = 2.998 \times 10^8 \, m s^{-1}$$
$$G = 6.67 \times 10^{-11} \, m^3 kg^{-1} s^{-2}$$
$$\omega = 7.292 1159 \times 10^{-5} \, rad \, s^{-1}$$

So

$$\boxed{\Omega = 1.52 \times 10^{-14} \, rad \, s^{-1}} \quad -(8)$$

which is the magnitude of the Earth's gravitomagnetic field. Now use:

$$1 \text{ year} = 3.156 \times 10^7 \text{ seconds} \quad -(9)$$
$$1 \text{ radian} = 2.06265 \times 10^5 \text{ arcseconds} \quad -(10)$$

To find that:

3)

$$\Omega = 98.8 \text{ milliarcseconds a year} \quad -(11)$$

So the Lense Thirring precession frequency is:

$$\boxed{\Omega_{LT} = 49.4 \text{ milliarcseconds a year}} \quad -(12)$$

The NASA / Stanford experimental value is:

$$\Omega(exp) = (40.9 \pm 7.8) \text{ milliarcseconds a year} \quad -(13)$$

with an uncertainty of 19%.

The theoretical result in Eq. (12) is over-estimated because it has been assumed that:

$$\underline{\omega} \perp \underline{r} \quad -(14)$$

and this may not be quite accurate. It is known that the earth is not a perfect sphere for example.

The analogue of eq. (1) in magnetostatics is:

$$\underline{B} = \frac{\mu_0}{4\pi r^3}\left(\underline{m} - 3\underline{n}\left(\underline{m}\cdot\underline{n}\right)\right) \quad -(15)$$

where $\underline{m}$ is the magnetic dipole moment:

$$\underline{m} = -\frac{e}{2m}\underline{L} \quad -(16)$$

and where $-e$ is the charge on the electron and m its mass. Here $\underline{L}$ is the orbital angular momentum of the electron.

The gravitomagnetic permeability is:

$$\mu_{0GR} = \frac{4\pi G}{c^2} \quad -(17)$$

So:

$$\underline{\Omega} = \frac{G}{r^3 c^2}\left(\underline{m}_g - 3\underline{n}\left(\underline{m}_g \cdot \underline{n}\right)\right) \quad -(18)$$

here the gravitomagnetic dipole moment is defined in analogy to eq. (11) by replacing $-e$ by m. So

$$\underline{m}_g = \frac{1}{2}\underline{L} \quad -(19)$$

which incorporates the factor $\frac{1}{2}$ needed to generate the Larmor precession, which is the Lense Thirring precession. The angular momentum $\underline{L}$ is that of the earth regarded as a sphere:

$$\underline{L} = \frac{2}{5} M R^2 \underline{\omega} \quad -(20)$$

So

$$\Omega = \frac{MGR^2\omega}{5c^2r^3} \quad -(21)$$

$$= 49.4 \text{ milliarcseconds per year}$$

Q.E.D.