345(1): Geodetic Precession from the ECE2 Gravitational Field Equations

Consider the gravitomagnetic field of a sphere in the dipole approximation:

$$\underline{\Omega} = \frac{2G}{c^2 r^3}\left(\underline{L} - 3\left(\underline{L}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right)$$

$$= \frac{2G}{c^2 r^3}\left(1 - 3\left(\cancel{\underline{k}}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right)\underline{L} \quad -(1)$$

Here $\underline{L}$ is an angular momentum:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(3)$$

generated by a frame rotation at an orbital linear velocity of $\underline{v}$. The distance $\underline{r}$ is that from the centre of the sphere to a magnetic dipole moment such as the gyroscope on Gravity Probe B. In close analogy with the Lense Thirring effect, $\underline{\Omega}$ is the gravitomagnetic field of the sun, but in the geodetic effect the sun is a stationary mass $\underline{M}$. The earth of mass m rotates around the sun. This is an active rotation in a fixed coordinate system. This is equivalent to a passive rotation of the coordinate frame.

1) A gravitomagnetic torque is generated between the field $\underline{\Omega}$ and the earth's gravitomagnetic dipole moment $\underline{m}$:

$$\underline{T_q} = \underline{m} \times \underline{\Omega} \quad -(4)$$

resulting in the Larmor precession frequency:

$$\omega_L = \frac{\Omega}{2} \quad -(5)$$

In analogy with the Lense Thirring effect it has been assumed that the effective Landé factor is unity. So

$$\omega_L = \frac{MGv}{c^2 r^2}\left(1 - 3\left(\underline{k}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right) \quad -(6)$$

where it has been assumed that

$$\underline{L} = Mrv\,\underline{k} \quad -(7)$$

the velocity of the spinning frame is that of the earth in orbit around the sun, and r is the distance from the sun to the earth. We have:

$$r = 6.957 \times 10^9 \text{ metres} \quad -(8)$$

$$v = 2.978473 \times 10^4 \text{ m s}^{-1} \quad -(9)$$

$$\frac{MG}{c^2} = 1.475 \times 10^3 \text{ metres} \quad -(10)$$

so
$$\omega_L = 9.079\left(1 - 3\left(\underline{k}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right) \times 10^{-13}$$

radians per second.

$$= 2.8651 \times 10^{-4}\left(1 - 3\left(\underline{k}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right) \text{ radians per year}$$

$$= 59.10\left(\underline{1} - 3\left(\underline{k}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right) \quad \text{— (14)}$$

arc seconds per year.

The experimental result is:

$$\omega(\text{obs}) = 6.6144 \text{ arc seconds per year}$$

by gravity probe B. This means that

$$59.10\left(\underline{1} - 3\left(\underline{k}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r}\right) = 6.6144$$

and

$$3\left(\underline{k}\cdot\frac{\underline{r}}{r}\right)\frac{\underline{r}}{r} = \underline{1} - \frac{6.6144}{59.10} \quad \text{— (15)}$$

$$= 0.888$$

Therefore the geodetic effect is described exactly with the geometry defined in eq. (15).