345(6): Theory of the Geodetic Effect in ECE2 Gravitomagnetism

Consider the magnetic field of a current loop in the dipole approximation (Jackson 5.56, 3rd Ed.):

$$\underline{B} = \frac{\mu_0}{4\pi r^3}\left(3\underline{n}\left(\underline{n}\cdot\underline{m}\right)-\underline{m}\right) \quad -(1)$$

at a point r from a magnetic dipole moment $\underline{m}$ generated by a current loop. The potential is:

$$\underline{A} = \frac{\mu_0}{4\pi r^3}\,\underline{m}\times\underline{r} \quad -(2)$$

and Jackson assumes that:

$$\underline{B} = \underline{\nabla}\times\underline{A} \quad -(3)$$

In ECE2 gravitomagnetism, the equivalent of Eq. (1) is:

$$\underline{\Omega} = \frac{G}{2c^2r^3}\left(3\underline{n}\left(\underline{n}\cdot\underline{L}\right)-\underline{L}\right) - (4)$$

where the gravitomagnetic dipole moment is:

$$\underline{m}_{gr} = \frac{1}{2}\underline{L} \quad -(5)$$

where $\underline{L}$ is an orbital angular momentum. In eqs. (1) and (4):

$$\underline{n} = \frac{\underline{r}}{r}. \quad -(6)$$

If the Earth is considered to be spinning about axis $\underline{k}$, then:

$$\underline{L} = \underline{r}\times\underline{p} = MR^2\omega\,\underline{k} \quad -(7)$$

where R is the radius of the earth, M is its mass, and ω its angular velocity.

2) In this example:

$$\underline{n} \cdot \underline{m} = 0 \quad -(8)$$

because $\underline{r}$ is perpendicular to $\underline{k}$. In general:

$$\underline{r} = X \underline{i} + Y \underline{j} + Z \underline{k}. \quad -(9)$$

Gravity Probe B orbited in the XZ plane at 90° to the equator in a polar orbit. It orbited once every 90 minutes. So:

$$\underline{L}(GPB) = m r^2 \omega \underline{j} \quad -(9)$$

where m is the mass of the spacecraft, r is the distance from the centre of the earth to the spacecraft, and:

$$\omega = \frac{2\pi}{90 \times 60} = \frac{2\pi}{5400} \text{ rad s}^{-1} \quad -(10)$$

Eq. (9) is true from the point of view of a reference frame centred at the earth's centre. The spacecraft generates the angular momentum (9).

From the point of view of a frame of reference fixed on the spacecraft, the earth appears to be in a spinning frame of reference, and generates the angular momentum:

$$\underline{L}(\text{earth}) = \underline{M} r^2 \omega \underline{j}. \quad -(11)$$

The earth is a current loop of mass, which generates the gravitomagnetic field:

$$\underline{\Omega}(\text{earth}) = \frac{G}{2c^2 r^3}\left(3\underline{n}(\underline{n} \cdot \underline{L}) - \underline{L}\right) \quad -(12)$$

$$= \frac{MGr^2\omega}{2c^2r^3}\left(3\underline{n}(\underline{n}\cdot\underline{j}) - \underline{j}\right)$$

$$= \frac{MG\omega}{2c^2r}\left(\frac{3\underline{r}}{r}\left(\frac{\underline{r}}{r}\cdot\underline{j}\right) - \underline{j}\right) \quad -(13)$$

Here

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(14)$$

and

$$r^2 = X^2 + Y^2 + Z^2. \quad -(15)$$

So:

$$\underline{\Omega} = \frac{MG\omega}{2c^2r}\left(\frac{3Y(X\underline{i} + Y\underline{j} + Z\underline{k})}{X^2+Y^2+Z^2} - \underline{j}\right) \quad -(16)$$

for any point at the earth's surface. The gravitomagnetic field $\underline{\Omega}$ generates a torque with the gyroscope of Gravity Probe B. The gyroscope is a gravitomagnetic dipole moment. Therefore:

$$\underline{\Omega} = \frac{MG\omega}{2c^2r}\left(\frac{3XY\underline{i} + (3Y^2 - r^2)\underline{j} + 3YZ\underline{k}}{r^2}\right) \quad -(17)$$

Its magnitude is:

$$\Omega = \frac{MG\omega}{2c^2r}\left(\frac{\left(9X^2Y^2 + 9Y^2Z^2 + (3Y^2-r^2)^2\right)^{1/2}}{r^2}\right)$$

$$= 6.6144 \text{ arc seconds a year (observed)}$$

$$= 1.016 \times 10^{-12} \text{ rad s}^{-1} \quad -(18) \text{ (observed)}$$

4) Now use:

$$M = 5.98 \times 10^{24} \ \text{kg}$$
$$G = 6.67 \times 10^{-11} \ \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$$
$$c = 2.998 \times 10^{8} \ \text{m s}^{-1}$$
$$T = 5.4 \times 10^{3} \ \text{sec}$$

to find:

$$\frac{MG\omega}{2c^2 r} = \frac{MG\pi}{c^2 r T} = 3.678 \times 10^{-13} \ \text{rad s}^{-1} \qquad (19)$$

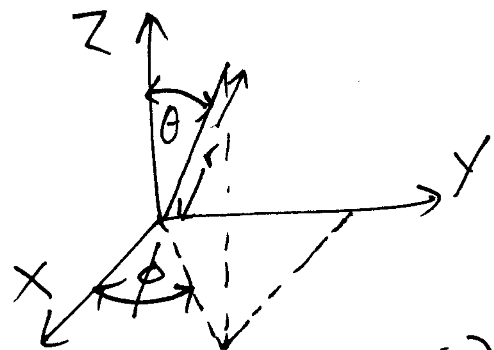
So:

$$x = \left\langle \frac{\left(9x^2y^2 + (3y^2 - r^2)^2 + 9y^2z^2\right)^{1/2}}{r^2} \right\rangle \qquad (20)$$

$$= 0.393$$

In spherical polar coordinates:

$$X = r\sin\theta\cos\phi$$
$$Y = r\sin\theta\sin\phi$$
$$Z = r\cos\theta$$



$$\text{so } x = \left\langle \left(9(\sin^2\theta\sin\phi\cos\phi)^2 + 9\sin^2\theta\cos^2\theta\sin^2\phi + (3\sin^2\theta\sin^2\phi - 1)^2\right)^{1/2} \right\rangle \qquad (21)$$

$$= \left\langle \left(9\left(\sin^4\theta\sin^2\phi\cos^2\phi + \sin^2\theta\cos^2\theta\sin^2\phi\right) + (3\sin^2\theta\sin^2\phi - 1)^2\right)^{1/2} \right\rangle \qquad (22)$$

5) Note that for

$$\phi = 0. \quad -(23)$$

then

$$x = 1, \quad -(24)$$

$$\theta = \pi/2 \quad -(25)$$

and for

$$x = 1 \quad -(26)$$

Using computer algebra, it is possible to choose values of θ and ϕ so that the result (20) is obtained exactly. This provides a plausible explanation of the geodetic effect, using the same starting equation (1) as the Lense-Thirring effect.