~~343~~ 344 (2): Development of Gravitomagnetism

Consider the Lorentz force law of gravitomagnetism in the non relativistic limit:

$$\underline{\Omega} = \frac{1}{c^2} \underline{v} \times \underline{g} \quad -(1)$$

as in previous ECE2 papers. Here $\underline{\Omega}$ is the gravitomagnetic field, $\underline{v}$ the velocity and $\underline{g}$ the acceleration due to gravity. This is the analogue of:

$$\underline{B} = \frac{1}{c^2} \underline{v} \times \underline{E} \quad -(2)$$

In the non-relativistic limit:

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2} \underline{e}_r \quad -(3)$$

so:

$$g = \frac{dv}{dt} = -\frac{d\Phi}{dr} \quad -(4)$$

where

$$\Phi = -\frac{MG}{r} \quad -(5)$$

is the gravitational potential.

So:

$$g = \frac{dv}{dt} = -\frac{MG}{r^2} \quad -(6)$$

Therefore:

$$v = -\int \frac{MG}{r^2} dt \quad -(7)$$

$$= \frac{dr}{dt},$$

and

$$v = -\int \frac{MG}{v r^2} dr \quad -(8)$$

For constant v,

$$v^2 = -MG \int \frac{dr}{r^2} = \frac{MG}{r} \quad -(9)$$

2) Therefore $$\underline{g} = -\frac{v^2}{r}\underline{e}_r \quad -(10)$$

an expression which gives the gravitational red shift and the equivalence principle.

From eqs. (1) and (10), the magnitude of the gravitomagnetic field is:

$$\Omega = \frac{v^3}{c^2 r} \quad -(11)$$

This can be expressed as:

$$\Omega = \left(\frac{v}{c}\right)^2 \omega \quad -(12)$$

where

$$v = \omega r \quad -(13)$$

for a circular orbit.

The gravitomagnetic Ampère Law is:

$$\underline{\nabla} \times \underline{\Omega} = \mu_{0g} \underline{J}_m \quad -(14)$$

where the gravitomagnetic permeability is:

$$\mu_{0g} = \frac{4\pi G}{c^2} \quad -(15)$$

The integral form of eq. (14) is:

$$\oint \underline{\Omega} \cdot d\underline{l} = \mu_{0g} \int_S \underline{J}_m \cdot \underline{n}\, dA \quad -(16)$$

$$= \mu_{0g} I_g$$

where the gravitational current for orbital motion is:

3)

$$I_g = \frac{2c^2 r \Omega}{4\pi G} \quad -(17)$$

Using eqs. (12), (13) and (17):

$$I_g = \frac{v^3}{2\pi G} \quad -(18) \quad \text{in } kg\,s^{-1}$$

which is the gravitational current for orbital motion of a mass m. A circular orbit has been used, but any type of orbit can be used.

The gravitomagnetic dipole moment is:

$$\mu_g = I_g A \quad -(19)$$

where for a circular orbit: $A = \pi r^2$.

The dipole moment has the dimensions of orbital angular momentum. Therefore:

$$\mu_g = \frac{v^3 r^2}{2G} \quad -(20)$$

The gravitomagnetic Faraday law of induction is:

$$\underline{\nabla} \times \underline{g} + \frac{\partial \underline{\Omega}}{\partial t} = \underline{0} \quad -(21)$$

from which the gravitomotive force is:

$$\mathcal{E}_g = -\frac{d\Phi_g}{dt} \quad -(22)$$

where Φ_g is the gravitomagnetic flux:

$$\Phi_g = \Omega A \quad -(23)$$

This is in direct analogy with the electromagnetic:

4)

$$F = \int_S \underline{B} \cdot \underline{n} \, dA \quad - (24)$$

and

$$\mathcal{E}_m = -\frac{dF}{dt} = \oint \underline{E}' \cdot d\underline{\ell} \quad - (25)$$

From eqs. (11) and (23):

$$\phi_g = \frac{\pi}{c^2} v^3 r \quad - (26)$$

in units of $m^2 s^{-1}$. Therefore the gravitomotive force is:

$$\mathcal{E}_g = -\frac{\pi}{c^2} \frac{d}{dt}\left(v^3 r\right) \quad - (27)$$

$$= -\frac{\pi v^4}{c^2}$$

for constant v as in a circular orbit, and using:

$$v = \frac{dr}{dt} \quad - (28)$$

Therefore:

$$\left| \frac{\phi_g}{2\pi} \right| = \frac{v^4}{2c^2} \quad - (29)$$

The gravitational Ohm law is:

$$V_g = R_g I_g \quad - (30)$$

where R_g is the gravitational resistance experienced by a moving mass due to the gravitational voltage V_g, where:

$$I_g = \frac{v^3}{2\pi G}, \quad \phi_g = \frac{\pi}{c^2} v^3 r \quad - (31)$$

5) Therefore the gravitational resistance is:

$$R_g = \frac{\pi G}{c^2} v = \frac{\mu_{og}}{4} v \quad -(32)$$

The gravitational power is:

$$P_g = I_g V_g = \frac{v^7}{4\pi G c^2} \quad -(33)$$

and this gives gravitational radiation.

The energy of interaction of $\underline{\mu_g}$ and $\underline{\Omega}$ is:

$$E_{int} = -\underline{\mu_g} \cdot \underline{\Omega} = -\frac{v^6 r}{2Gc^2} \quad -(34)$$

For the Earth this is:

$$E_{int} = -8.69 \times 10^{30} \; J. \quad -(35)$$

The Earth's total heat content (Arbab 2010) is 3×10^{31} J, and the Earth's rotational energy is 2.1×10^{29} J. The Earth's gravitoelectric power is 2.753×10^{23} watts, which is the energy consumed per year, converted to heat inside the earth.

The torque between $\underline{\mu_g}$ and $\underline{\Omega}$ is:

$$\underline{Tq} = \underline{\mu_g} \times \underline{\Omega} \quad -(36)$$

which makes the orbiting m precess with a gravitomagnetic Larmor frequency:

b) $$\omega_{Larmor} = \frac{m}{2m} g_{eff}\, \Omega = \frac{g_{eff}}{2}\, \Omega \quad -(37)$$

where g_{eff} is the gravitomagnetic Landé factor.

In general:

$$\Omega_g = \frac{g_{eff}}{2}\, \Omega \quad -(38)$$

and in calculating the Lense Thirring effect in 344(1) we used

$$g_{eff} = 1 \quad -(39)$$

The gravitomagnetic dipole moment is eq. (20) for the units of angular momentum, and can be written as:

$$\underline{\mu_g} = \frac{g_{eff}}{2}\, \underline{L} \quad -(40)$$

The orbiting mass m precesses with Larmor frequency:

$$\omega_{Larmor} = \Omega_g \quad -(41)$$

The magnitude of the torque is:

$$Tq = \Omega_g L \quad -(42)$$

From fundamental kinematics:

$$\underline{Tq} = \frac{d\underline{L}}{dt} = \underline{\Omega_g} \times \underline{L} \quad -(43)$$

Therefore gravitomagnetic theory shows that the precession of the perihelia is a Larmor precession.