342(2): Law of Gravitation for Finite Dimensions

For point particles, the gravitational attraction between m and M, separated by a distance r, is described by the force law:

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2}\underline{e}_r \quad -(1)$$

but for an object of finite dimensions:

$$\underline{F} = m\underline{g} = -mG\int_V \frac{\rho(\underline{r}')\,\underline{e}_r}{r^2}\,dV' \quad -(2)$$

where $\rho(\underline{r}')$ is the mass density and dV' the volume element at $\underline{r}'$

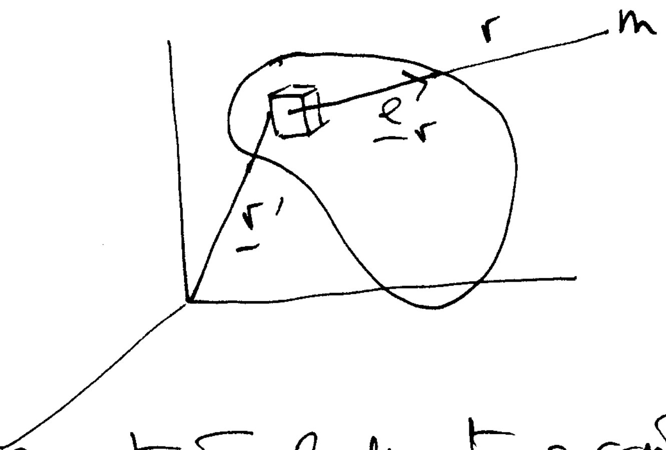


Fig (1)

The gravitational potential due to a continuous distribution of matter is:

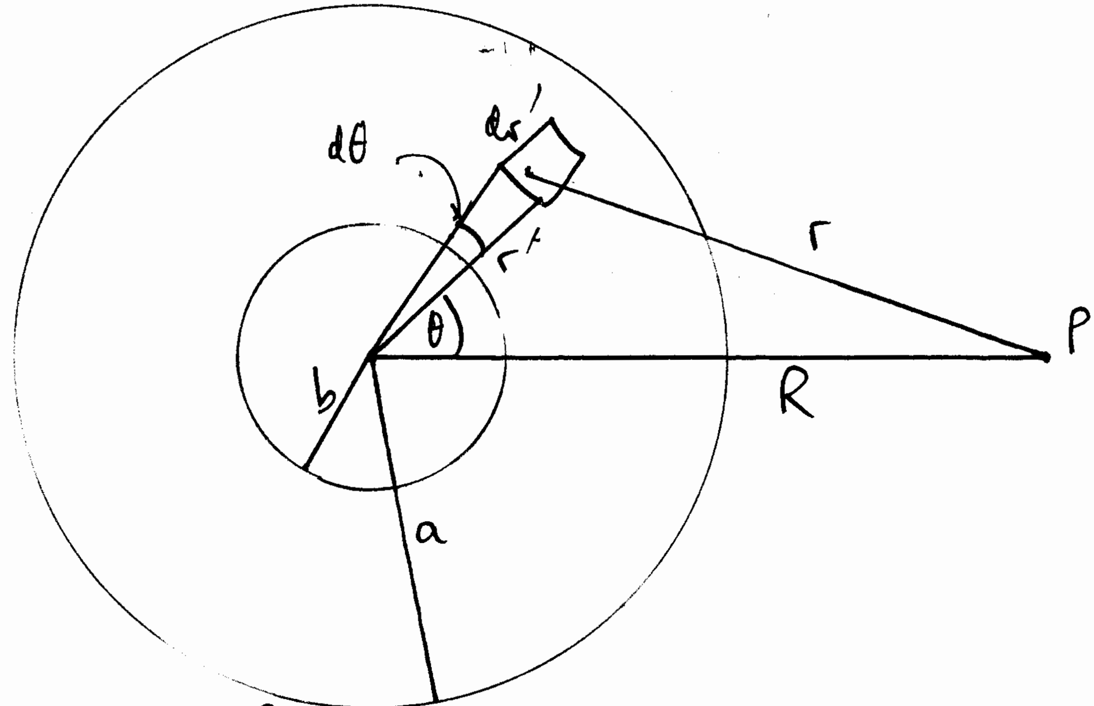
$$\phi = -G\int_V \frac{\rho(\underline{r}')}{r}\,dV' \quad -(3)$$

As in note 342(1) the continuous distribution of matter is made up of gravitons.

To explain this theory calculate the gravitational potential inside and outside a spherical shell of inner radius b and outer radius a.

2)

Fig(2)



The potential in this case is:

$$\phi = -G \int_V \frac{\rho(r')}{r} \, dV' \quad -(4)$$

$$= -2\pi \rho G \int_b^a r'^2 dr' \int_0^\pi \frac{\sin\theta}{r} \, d\theta$$

where there is a spherically symmetric mass distribution:

$$\rho = \rho(r') \quad -(5)$$

In Fig(2): $r^2 = r'^2 + R^2 - 2r'R\cos\theta. \quad -(6)$

Note that R is a constant for a given r', so:

$$2r\,dr = 2r'R\sin\theta\, d\theta \quad -(7)$$

by differentiating eq. (6). Therefore:

$$\frac{\sin\theta\, d\theta}{r} = \frac{dr}{r'R} \quad -(8)$$

3) It follows that:

$$\phi = -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{r_{min}}^{r_{max}} dr \qquad -(9)$$

If P is outside the shell then:

$$\phi(R>a) = -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{R-r'}^{R+r'} dr$$

$$= -\frac{4\pi\rho G}{R}\int_a^b r'^2 dr'$$

$$= -\frac{4}{3}\frac{\pi\rho G}{R}\left(a^3-b^3\right) \qquad -(10)$$

The mass of the spherical shell is:

$$M = \frac{4}{3}\pi\rho\left(a^3-b^3\right) \qquad -(11)$$

so

$$\boxed{\phi(R>a) = -\frac{MG}{R}} \qquad -(12)$$

From Note 342(1) the mass density is:

$$\rho = \sum_{i=1}^{n} m_i\,\delta(\underline{r}) \qquad -(13)$$

i.e. is a linear superposition of gravitons of mass $\underline{m}$.

Here $\delta(\underline{r})$ is the Dirac Delta function.

The usual potential used in Eq. (12) is

4) calculating the deflection of the orbit of a mass by an object such as the earth or sun. Note that the spherical object behaves as a point particle of mass M concentrated at its centre.

This was probably first realized by Newton in work from 1665 to 1687.

The total acceleration due to gravity is a linear superposition:

$$\underline{g}(r) = -G \sum_{i=1}^{n} \frac{m_i}{r^2} \underline{e}_r \quad -(14)$$

of the acceleration due to each graviton.

From eqs. (13) and (14):

$$\underline{g}(r) = -G \int \frac{\rho(\underline{r}')}{r^2} \underline{e}_r \, dV' \quad -(15)$$

If there is only one graviton, the theory reduced to the gravitational attraction of an orbiting mass to one graviton. If the orbiting mass is a photon then the theory is the attraction of one photon to one graviton.

In general, the theory is the attraction of n photons in a beam of light to m gravitons. The n photons have a mass m, in a mass M. Then

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2} \underline{e}_r \quad -(16)$$

so: