342(8): The Precessional Equation

The fundamental equation is:

$$v^2 = \frac{L^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2_{rel}\right) \quad -(1)$$

where v is the relativistic velocity, L is the relativistic angular momentum, a constant of motion, and where $(dr/d\theta)_{rel}$ is the relativistic orbit.

By experiment the relativistic orbit is a precessing ellipse, which can be represented by

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(2)$$

where

$$x = 1 - \frac{3MG}{c^2\alpha} \quad -(3)$$

From eq. (2):

$$v^2 = \frac{L^2}{m^2 r^2}\left(\frac{1}{r^2} + \frac{x^2\epsilon^2}{\alpha^2}\sin^2(x\theta)\right) \quad -(4)$$

By definition:

$$v^2 = \frac{v_N^2}{1 - \frac{v_N^2}{c^2}} \quad -(5)$$

where v_N, denoting the Newtonian orbital velocity, 2)

is defined by a conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta}, \quad -(6)$$

So

$$v_N^2 = \frac{L_0^2}{m^2 r^2}\left(\frac{1}{r^2} + \frac{\epsilon^2}{\alpha^2}\sin^2\theta\right). \quad -(7)$$

Here:

$$L = \gamma L_0 \quad -(8)$$

where γ is the Lorentz factor:

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(9)$$

It follows that:

$$L^2\left(\frac{1}{r^2} + \frac{x^2\epsilon^2}{\alpha^2}\sin^2(x\theta)\right) \quad -(10)$$

$$= \frac{L_0^2\left(\frac{1}{r^2} + \frac{\epsilon^2}{\alpha^2}\sin^2\theta\right)}{1 - \left(\frac{L_0}{mrc}\right)^2\left(\frac{1}{r^2} + \frac{\epsilon^2}{\alpha^2}\sin^2\theta\right)}$$

where

$$x = 1 - \frac{3mG}{c^2\alpha} \quad -(11)$$

and

3) and

$$\alpha = a(1-\epsilon^2) \quad - (12)$$

where a is the semi-major axis of the ellipse. Eqs. (11) and (12) are known experimentally to be valid with great precision.

From eq. (8):

$$\left(\frac{L}{L_0}\right)^2 = \gamma^2 \quad - (13)$$

In eq. (10), L^2 is a constant of motion. From eq. (6):

$$L_0^2 = mMG\alpha \quad - (14)$$

$$= mMGa(1-\epsilon^2)$$

Therefore:

$$L^2\left(\frac{1}{r^2} + \frac{x^2\epsilon^2}{a^2(1-\epsilon^2)^2}\sin^2(x\theta)\right) \quad - (15)$$

$$= \frac{mMGa(1-\epsilon^2)\left(\frac{1}{r^2} + \frac{\epsilon^2}{a^2(1-\epsilon^2)^2}\sin^2\theta\right)}{1 - \frac{MGa^2}{r^2c^2}(1-\epsilon^2)\left(\frac{1}{r^2} + \frac{\epsilon^2}{a^2(1-\epsilon^2)^2}\sin^2\theta\right)}$$

where

$$x = 1 - \frac{3MG}{c^2a(1-\epsilon^2)} \quad - (16)$$

4) In eq. (15) L is a constant. So eq. (15) gives a relation between r and θ, and r can be solved in terms of θ and graphed. This function is the one required for ECE2 to produce the precisely correct orbital precession. ECE2 also produces the precisely correct angle of deflection.

More generally, the precessing ellipse (2) can be expressed as:

$$r = \frac{\alpha}{1 + \epsilon \cos(\theta - \Delta\theta)} \quad -(17)$$

but eqs. (2) and (3) are valid experimentally to great precision. This appears to be true for all precessing orbits hitherto observed.