342(5): Details of 342(4), and New Orbit

It is useful to provide all details of this calculation for students, and to check them. It starts with the expression for the Newtonian velocity in plane polar coordinates:

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(1)$$

The relevant orbit is the conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta}. \quad -(2)$$

Now use:

$$\frac{dr}{dt} = \left(\frac{dr}{d\theta}\right)\left(\frac{d\theta}{dt}\right) \quad -(3)$$

so

$$v_N^2 = \left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(4)$$

$$= \left(\frac{d\theta}{dt}\right)^2\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)$$

From a Lagrangian analysis:

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(5)$$

where L is the conserved angular momentum. Also:

$$\frac{dr}{d\theta} = \frac{\epsilon r^2\sin\theta}{\alpha} \quad -(6)$$

So

$$v_N^2 = \frac{L^2}{m^2r^4}\left(r^2 + r^4\left(\frac{\epsilon}{\alpha}\right)^2\sin^2\theta\right) \quad -(7)$$

$$= \frac{L^2}{m^2}\left(\frac{1}{r^2} + \left(\frac{\epsilon}{\alpha}\right)^2\sin^2\theta\right)$$

). From eq. (2):

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(8)$$

and

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2. \quad -(9)$$

So:

$$v_N^2 = \frac{L^2}{m^2}\left(\frac{1}{r^2} + \left(\frac{\epsilon}{\alpha}\right)^2\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha^2}{r^2} - \frac{2\alpha}{r} + 1\right)\right)\right)$$

$$= \frac{L^2}{m^2}\left(\frac{1}{r^2} + \left(\frac{\epsilon}{\alpha}\right)^2 - \frac{1}{r^2} + \frac{2}{\alpha r} - \frac{1}{\alpha^2}\right)$$

$$= \frac{L^2}{m^2}\left(\frac{\epsilon^2 - 1}{\alpha^2} + \frac{2}{\alpha r}\right) \quad -(10)$$

Now use:

$$L^2 = \alpha m^2 \underline{M} G \quad -(11)$$

so

$$v_N^2 = \underline{M} G\left(\frac{\epsilon^2 - 1}{\alpha} + \frac{2}{r}\right) \quad -(12)$$

Using:

$$a = \frac{\alpha}{1 - \epsilon^2} \quad -(13)$$

we have

$$v_N^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(14)$$

<u>This is Eq. (7.72)</u> <u>of Marion and</u> <u>Thornton.</u>

For the hyperbola:

$$\epsilon > 1 \quad -(15)$$

and for a photon grazing the sun:

$$\epsilon \gg 1 \,. \quad -(16)$$

At closest approach:

$$\cos\theta = 1 \,, \quad r = R_0 \quad -(17)$$

so

$$R_0 = \frac{\alpha}{1+\epsilon} \quad -(18)$$

Therefore at closest approach:

$$v_N^2 = MG\left(\frac{\epsilon^2-1}{R_0(1+\epsilon)} + \frac{2}{R_0}\right)$$

$$= \frac{MG}{R_0}\left(\frac{(\epsilon+1)(\epsilon-1)}{\epsilon+1} + 2\right)$$

$$= \frac{MG}{R_0}\left(\epsilon + 1\right)$$

$$\sim \frac{MG\epsilon}{R_0} \quad -(19)$$

So

$$\epsilon \sim \frac{R_0 v_N^2}{MG} \quad -(20)$$

The total deflection is the angle between the asymptotes of a hyperbola:

$$\Delta\phi = 2\sin^{-1}\frac{1}{\epsilon} = 2\tan^{-1}\frac{b}{a} \quad -(21)$$

Therefore:
$$\sin\left(\frac{\Delta\phi}{2}\right) = \frac{1}{\epsilon} \quad -(22)$$
$$\sim \frac{\Delta\phi}{2}$$
for small $\Delta\phi$. So:
$$\Delta\phi = \frac{2MG}{R_0 v_N^2} \quad -(23)$$
The experimental result is:
$$\Delta\phi = \frac{4MG}{R_0 c^2}. \quad -(24)$$
Eq. (24) is the result of the equation for the relativistic velocity:
$$v^2 = \gamma^2 v_N^2 = \left(1 - \frac{v_N^2}{c^2}\right)^{-1} v_N^2 \quad -(25)$$
so
$$v^2 = \frac{v_N^2}{1 - \frac{v_N^2}{c^2}} \quad -(26)$$
and
$$v_N^2 = \frac{v^2}{1 + \frac{v^2}{c^2}} \quad -(27)$$
The observable velocity is always $\underline{v}$. The Newtonian velocity is the limit:
$$\underline{v} \xrightarrow[v_N \ll c]{} \underline{v}_N \quad -(28)$$

5)

By definition the relativistic velocity of a particle of mass m orbiting a mass M is:

$$v^2 = \frac{v_N^2}{1 - \frac{v_N^2}{c^2}} = \frac{\frac{L^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)}{1 - \frac{L^2}{c^2 m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)} \quad -(29)$$

in which $dr/d\theta$ is defined by the conic section (2).

However, as in UFT328, it is known that the true orbit is a precessing ellipse, a conic section, given by the simultaneous solution of the hamiltonian and lagrangian of special relativity. Denote this precessing conic section by:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta_1} \quad -(30)$$

where θ_1 is a function of θ to be determined from:

$$v^2 = \frac{L^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta_1}\right)^2\right) = \frac{\frac{L^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)}{1 - \frac{L^2}{c^2 m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)}$$

$-(31)$

6) Using:

$$v_N^2 = \frac{L^2}{m^2}\left(\frac{1}{r^2} + \left(\frac{\epsilon}{\alpha}\right)^2 \sin^2\theta\right) \quad -(32)$$

$$= \frac{L^2}{m^2}\left(\left(\frac{1+\epsilon\cos\theta}{\alpha}\right)^2 + \left(\frac{\epsilon}{\alpha}\right)^2 \sin^2\theta\right)$$

$$= \frac{L^2}{m^2\alpha^2}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta + 2\epsilon\cos\theta\right)$$

It follows that:

$$v^2 = \frac{\frac{L^2}{m^2\alpha^2}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta + 2\epsilon\cos\theta\right)}{1 - \frac{L^2}{m^2\alpha^2c^2}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta + 2\epsilon\cos\theta\right)} \quad -(33)$$

Now use:

$$L^2 = m^2 M G \alpha \quad -(34)$$

$$v^2 = \frac{\frac{MG}{\alpha}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta + 2\epsilon\cos\theta\right)}{1 - \frac{MG}{\alpha c^2}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta + 2\epsilon\cos\theta\right)} \quad -(34)$$

7) This is the relativistic velocity of an object of mass m orbiting a mass M. The new relativistic velocity is given by:

$$v_N^2 = \frac{MG}{\alpha}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta + 2\epsilon\cos\theta\right) - (35)$$

The effect of the relativistic correction can be seen by graphing eqs. (34) and (35).

However, it is known from UFT328 that the relativistic correction produces a precessing ellipse. The latter is considered most generally to be the function (30) of the coordinate θ_1. This is a conic section in the coordinate system (r, θ_1). In this coordinate system the relativistic velocity is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta_1}{dt}\right)^2 - (36)$$

$$= \frac{MG}{\alpha}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta_1 + 2\epsilon\cos\theta_1\right)$$

From eqs. (34) and (36), a relation can be obtained between θ_1 and θ:

$$1+\epsilon^2+(1-\epsilon^2)\cos^2\theta_1+2\epsilon\cos\theta_1 \quad -(37)$$

$$= \frac{1+\epsilon^2+(1-\epsilon^2)\cos^2\theta+2\epsilon\cos\theta}{1-\frac{MG}{\alpha c^2}\left(1+\epsilon^2+(1-\epsilon^2)\cos^2\theta+2\epsilon\cos\theta\right)}$$

Since

$$\frac{MG}{\alpha c^2} \rightarrow 0 \quad -(38)$$

in the non-relativistic limit, it is seen that in this limit:

$$\theta_1 = \theta \quad -(39)$$

otherwise:

$$\theta_1 \neq \theta \quad -(40)$$

and the orbit is no longer a static conic section. This is a very important result of ECE2 special relativity. It agrees with the result of UFT328.

If

$$\theta = 2\pi, \quad -(41)$$

the angle for one complete orbit of the static conic section, eq. (37) simplifies to:

$$1+\epsilon^2+(1-\epsilon^2)\cos^2\theta_1+2\epsilon\cos\theta_1 = \frac{2(1+\epsilon)}{1-\frac{2MG}{\alpha c^2}(1+\epsilon)} \quad -(42)$$

and the angle θ_1 can be found analytically by solving the quadratic equation (42) with computer algebra.

More generally, $\cos\theta_1$ can be found in terms of $\cos\theta$ by solving eq. (37) by computer algebra. This gives the conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta_1} = \frac{\alpha}{1+\epsilon f(\cos\theta)} \quad -(43)$$

Finally the orbit $r(\theta)$ can be plotted in polar coordinates.