337(3): Lamb Shift and other Energy Shifts due to the Vacuum Scalar Potential ϕ

Consider the energy equation of ECE2 special relativity:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(1)$$

The effect of the vacuum scalar potential ϕ is given by the minimal prescription:

$$E \rightarrow E - e\phi . \quad -(2)$$

Therefore:

$$(E - e\phi)^2 = c^2 p^2 + m^2 c^4 \quad -(3)$$

i.e.

$$(E - e\phi - mc^2)(E - e\phi + mc^2) = c^2 p^2 \quad -(4)$$

and

$$E - e\phi - mc^2 = \frac{c^2 p^2}{E - e\phi + mc^2} . \quad -(5)$$

For the sake of argument use the Dirac approximation:

$$E = mc^2 . \quad -(6)$$

More accurately and correctly:

$$E = \gamma mc^2 . \quad -(7)$$

In the approximation (6):

$$E - e\phi - mc^2 = \frac{1}{2m} p^2 \left(1 - \frac{e\phi}{2mc^2} \right)^{-1} \quad -(8)$$

where p is the relativistic momentum.

On the classical level the ubiquitous vacuum changes the total relativistic energy by (7), and

$$\frac{p^2}{2m} \rightarrow \frac{p^2}{2m}\left(1 - \frac{e\phi}{2mc^2}\right)^{-1} \quad -(9)$$

The vacuum is always present, so eq. (8) is the true equation. In the hypothetical absence of ϕ, eq. (8) becomes

$$T = E - mc^2 = \frac{p^2}{2m} \quad -(10)$$

where T is the relativistic kinetic energy:

$$T = (\gamma - 1)mc^2 \quad -(11)$$

In the non-relativistic limit:

$$v << c, \quad -(12)$$

then

$$T = \left(\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} - 1\right)mc^2 \quad -(13)$$

$$\rightarrow \left(1 + \frac{v_0^2}{c^2} - 1\right)mc^2 = mv_0^2$$

so we recover the classical non-relativistic result:

$$mv_0^2 = \frac{p_0^2}{2m} \quad -(14)$$

Q.E.D

In the usual development it is assumed that:

3)

$$E - e\phi - mc^2 = \frac{1}{2m}\,\underline{\sigma}\cdot\underline{p}\left(1 - \frac{e\phi}{2mc^2}\right)^{-1}\underline{\sigma}\cdot\underline{p} \quad -(15)$$

in the SU(2) basis. Assuming:

$$e\phi \ll 2mc^2 \quad -(16)$$

then

$$E - e\phi - mc^2 \doteq \frac{1}{2m}\,\underline{\sigma}\cdot\underline{p}\left(1 + \frac{e\phi}{2mc^2}\right)\underline{\sigma}\cdot\underline{p} \quad -(17)$$

$$= \frac{1}{2m}\,\underline{\sigma}\cdot\underline{p}\;\underline{\sigma}\cdot\underline{p} + \underline{\sigma}\cdot\underline{p}\,\frac{e\phi}{4m^2c^2}\,\underline{\sigma}\cdot\underline{p}$$

$$:= H_1 + H_2$$

In order to observe the effect of ϕ experimentally it is necessary to quantize eq. (17) to produce effects in the hydrogen atom. There are two hamiltonians to consider:

$$H_1\psi = -\frac{i\hbar}{2m}\,\underline{\sigma}\cdot\underline{\nabla}\left(\underline{\sigma}\cdot\underline{p}\,\psi\right) \quad -(18)$$

and

$$H_2\psi = -\frac{ie\hbar}{4m^2c^2}\,\underline{\sigma}\cdot\underline{\nabla}\left(\phi\,\underline{\sigma}\cdot\underline{p}\,\psi\right) \quad -(19)$$

Eqs. (18) and (19) must be developed with the Leibniz Theorem and produce a large number of effects, few if any of which have been investigated. All these effects come from the vacuum or spacetime.

4) The relevant term containing the vacuum scalar potential ϕ is eq. (19), but it is also relevant to consider eq. (18). Using the Leibniz theorem:

$$\underline{\nabla}(\underline{\sigma}\cdot\underline{p}\,\psi) = \psi\,\underline{\nabla}(\underline{\sigma}\cdot\underline{p}) + \underline{\sigma}\cdot\underline{p}\,\underline{\nabla}\psi \quad -(20)$$

so

$$\underline{\sigma}\cdot\underline{\nabla}(\underline{\sigma}\cdot\underline{p}\,\psi) = (\underline{\sigma}\cdot\underline{\nabla}\,\underline{\sigma}\cdot\underline{p})\psi + \underline{\sigma}\cdot\underline{p}\,\underline{\sigma}\cdot\underline{\nabla}\psi \quad -(21)$$

Therefore:

$$H_1\psi = -\frac{i\hbar}{2m}\left(\underline{\sigma}\cdot\underline{p}\,\underline{\sigma}\cdot\underline{\nabla}\psi + \underline{\sigma}\cdot\underline{\nabla}\,\underline{\sigma}\cdot\underline{p}\,\psi\right) \quad -(22)$$

$$= \frac{p^2}{2m}\psi - \frac{i\hbar}{2m}\left(\underline{\nabla}\cdot\underline{p} + i\,\underline{\sigma}\cdot\underline{\nabla}\times\underline{p}\right)\psi$$

The physical part of eq. (22) is its real part, so:

$$\boxed{E_1 = \langle H_1\rangle = \frac{p^2}{2m} + \frac{\hbar}{2m}\,\underline{\sigma}\cdot\underline{\nabla}\times\underline{p}} \quad -(23)$$

Introducing quantization in the SU(2) basis produces a new kinetic energy term, the second term on the right hand side of eq. (23). This does not seem to have seen clearly realized in the literature. This term can be developed as follows:

). The orbital angular momentum is:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(24)$$

and it follows that:

$$\underline{r} \times \underline{L} = \underline{r} \times (\underline{r} \times \underline{p}) = \underline{r}(\underline{r} \cdot \underline{p}) - \underline{p}(\underline{r} \cdot \underline{r})$$

$$= \underline{r}(\underline{r} \cdot \underline{p}) - r^2 \underline{p} \quad -(25)$$

Therefore:

$$\underline{p} = \frac{1}{r^2}\left(\underline{r}(\underline{r} \cdot \underline{p}) - \underline{r} \times \underline{L}\right) \quad -(26)$$

$$= \frac{1}{r^2}\left(\underline{L} \times \underline{r} + \underline{r}(\underline{r} \cdot \underline{p})\right)$$

Therefore:

$$\underline{\nabla} \times \underline{p} = \underline{\nabla} \times \left(\frac{1}{r^2}\left(\underline{L} \times \underline{r} + \underline{r}(\underline{r} \cdot \underline{p})\right)\right). \quad -(27)$$

Now use the vector algebra:

$$\underline{\nabla} \times (U\underline{F}) = U\underline{\nabla} \times \underline{F} + \underline{\nabla} U \times \underline{F}, \quad -(28)$$

to find that:

$$\underline{\nabla} \times \left(\frac{1}{r^2}\underline{L} \times \underline{r}\right) = \frac{1}{r^2}\left(\underline{\nabla} \times (\underline{L} \times \underline{r}) + \underline{\nabla}\left(\frac{1}{r^2}\right) \times (\underline{L} \times \underline{r})\right) \quad -(29)$$

and

$$\underline{\nabla} \times \left(\frac{1}{r^2}\underline{r}(\underline{r} \cdot \underline{p})\right) = \frac{\underline{r} \cdot \underline{p}}{r^2}\, \underline{\nabla} \times \underline{r} + \underline{\nabla}\left(\frac{\underline{r} \cdot \underline{p}}{r^2}\right) \times \underline{r} \quad -(30)$$

From eqs. (27) to (30) it is clear that several

Hitherto unknown terms exist.

For the sake of simplicity, restrict attention to the first term only, to find that:

$$E_1 = \frac{p^2}{2m} + \frac{\hbar}{2m} \underline{\sigma} \cdot \frac{1}{r^2} \underline{\nabla} \times (\underline{L} \times \underline{r}) + \dots \quad -(31)$$

From vector algebra:

$$\underline{\nabla} \times (\underline{L} \times \underline{r}) = \underline{L} (\underline{\nabla} \cdot \underline{r}) - (\underline{\nabla} \cdot \underline{L}) \underline{r} + (\underline{r} \cdot \underline{\nabla}) \underline{L} - (\underline{L} \cdot \underline{\nabla}) \underline{r} \quad -(32)$$

in which:

$$\underline{\nabla} \cdot \underline{r} = 3. \quad -(33)$$

It follows that:

$$E_1 = \frac{p^2}{2m} + \frac{3\hbar}{2m} \frac{\underline{\sigma} \cdot \underline{L}}{r^2} + \dots \quad -(34)$$

This is a hitherto unknown spin-orbit term.

The spin angular momentum is:

$$\underline{S} = \frac{\hbar}{2} \underline{\sigma} \quad -(35)$$

So the new spin orbit hamiltonian is:

$$H_{so}\phi = \frac{3}{m}\frac{\underline{S}\cdot\underline{L}}{r^2}\phi \quad -(36)$$

The expectation value of $\mathcal{C}_{ij}$ term is:

$$E = \frac{3}{m}\left\langle \frac{\underline{S}\cdot\underline{L}}{r^2}\right\rangle$$

$$= \frac{3\hbar^2}{2m}\left(J(J+1) - L(L+1) - S(S+1)\right)\left\langle\frac{1}{r^2}\right\rangle \quad -(37)$$

where $\left\langle \frac{1}{r^2}\right\rangle = \int \phi^* \frac{1}{r^2}\phi \, d\tau \quad -(38)$

The observable effect of the vacuum scalar potential are worked out from:

$$H_2\phi = -\frac{ie\hbar}{4m^2c^2}\,\underline{\sigma}\cdot\underline{\nabla}\left(\phi\,\underline{\sigma}\cdot\underline{p}\,\phi\right) \quad -(39)$$

where:

$$\underline{\nabla}\left(\phi\,\underline{\sigma}\cdot\underline{p}\,\phi\right) = \underline{\nabla}\phi\left(\underline{\sigma}\cdot\underline{p}\,\phi\right) + \phi\,\underline{\nabla}\left(\underline{\sigma}\cdot\underline{p}\,\phi\right)$$

$$= \underline{\nabla}\phi\left(\underline{\sigma}\cdot\underline{p}\,\phi\right) + \phi\left(\underline{\nabla}\left(\underline{\sigma}\cdot\underline{p}\right)\right)\phi + \phi\left(\underline{\sigma}\cdot\underline{p}\right)\underline{\nabla}\phi \quad -(40)$$

So the relevant Hamiltonian is:

8)
$$H_2 \phi = -\frac{ie\hbar\phi}{4m^2c^2}\left(\underline{\sigma}\cdot\underline{\nabla}\,\underline{\sigma}\cdot\underline{p}\,\phi + \underline{\sigma}\cdot\underline{\nabla}\phi\,\underline{\sigma}\cdot\underline{p}\right)$$
$$-\frac{ie\hbar}{4m^2c^2}\left(\underline{\sigma}\cdot\underline{\nabla}\phi\,\underline{\sigma}\cdot\underline{p}\,\phi\right) \quad -(41)$$

The real and physical part of this hamiltonian is:
$$\mathrm{Real}\, H_2\phi = \frac{e\hbar}{4m^2c^2}\,\underline{\sigma}\cdot\left(\left(\underline{\nabla}\phi\times\underline{p}\right)\phi + \phi\,\underline{\nabla}\times\underline{p}\,\phi + \phi\,\underline{\nabla}\phi\times\underline{p}\right) \quad -(42)$$

For the sake of illustration and simplicity we consider the term:
$$E = \frac{e\hbar\phi}{4m^2c^2}\,\underline{\sigma}\cdot\underline{\nabla}\times\underline{p} + \cdots \quad -(43)$$
$$H\phi = \frac{3e\hbar\phi}{4m^2c^2r^2}\,\underline{\sigma}\cdot\underline{L}\,\phi + \cdots \quad -(44)$$
$$= \frac{3e\phi}{2m^2c^2r^2}\,\underline{S}\cdot\underline{L}\,\phi + \cdots$$

This is a contribution to the fine structure of the H atom with expectation value:

i)

$$\boxed{E = \frac{3e\phi\hbar^2}{4m^2c^2}\left(J(J+1) - L(L+1) - S(S+1)\right)\left\langle \frac{1}{r^2} \right\rangle} \quad -(45)$$

This effect is observable experimentally, and its derivation is the same in ECE2 and the standard physics. It is derived directly from eq. (3) with the Dirac approximation.
