337(1): The Lamb Shift from the AB Vacuum.

Consider the spin orbit term of eq. (26) of Note 336(5):

$$H_{LS} = \frac{1}{m(1+\gamma)} \underline{\sigma}\cdot(\underline{p} - e\underline{A}) \frac{e\phi}{(1+\gamma)mc^2} \underline{\sigma}\cdot(\underline{p} - e\underline{A}) \quad -(1)$$

in the same notation as note 336(5) Eq. (1) is:

$$H_{LS} = \frac{e}{m^2(1+\gamma)^2c^2} \underline{\sigma}\cdot(\underline{p} - e\underline{A})\phi\, \underline{\sigma}\cdot(\underline{p} - e\underline{A}) \quad -(2)$$

Eq. (2) is quantized as:

$$H_{LS}\psi = \frac{e}{m^2(1+\gamma)^2c^2} \underline{\sigma}\cdot(-i\hbar\underline{\nabla} - e\underline{A})\phi\, \underline{\sigma}\cdot(\underline{p} - e\underline{A})$$

$$= \frac{-i\hbar e}{m^2(1+\gamma)^2c^2} \underline{\sigma}\cdot\underline{\nabla}\phi\, \underline{\sigma}\cdot(\underline{p} - e\underline{A})\psi + \dots \quad -(3)$$

where

$$\underline{\nabla}\phi = \frac{e}{4\pi\epsilon_0}\frac{\underline{r}}{r^3} \quad -(4)$$

Therefore:

$$\mathrm{Re}\, H_{LS}\psi = \frac{\hbar e^2}{4\pi\epsilon_0 m^2c^2(1+\gamma)^2r^3} \underline{\sigma}\cdot\underline{r}\times(\underline{p} - e\underline{A}) \quad -(5)$$

where

$$\gamma = \frac{\hbar\omega}{mc^2} \quad -(6)$$

Eq. (5) gives the usual spin orbit term in

2) augmented by the Lamb shift term. The spin angular momentum of the electron is:

$$\underline{S} = \frac{\hbar}{2}\underline{\sigma} \quad -(7)$$

and the usual spin orbit term is:

$$Re H_{so}\phi = \frac{e^2}{2\pi\epsilon_0 m^2 c^2 (1+\gamma)^2 r^3}\underline{S}\cdot\underline{L}\phi \quad -(8)$$

where

$$\underline{L} = \underline{r}\times\underline{p} \quad -(9)$$

The expectation value from eq. (8) is: $-(10)$

$$\langle H_{so}\rangle = \frac{e^2\hbar^2}{4\pi\epsilon_0 c^2 m^2 (1+\gamma)^2 a_0^3}\left(\frac{J(J+1)-L(L+1)-S(S+1)}{n^3 L(L+1/2)(L+1)}\right)$$

where

$$J = L+S, \ldots, |L-S| \quad -(11)$$

Here a_0 is the Bohr radius and n the principal quantum number. In the usual theory:

$$\gamma = 1 \quad -(12)$$

but more accurately eq. (11) must be used because eq. (12) produces the unphysical result:

$$H_0 = H - mc^2 =? \; 0 \quad -(13)$$

Eq. (11) describes the fine structure of atomic hydrogen H with the exception of the Lamb shift, observed by Retherford and Lamb in the mid forties.

3) The Lamb shift is described in this theory by:

$$H_{LS} = \frac{-e^2}{2\pi \epsilon_0 m^2 c^2 (1+\gamma)^2 r^3} \underline{S} \cdot \underline{L}_{vac} \quad -(14)$$

where the vacuum angular momentum is:

$$\underline{L}_{vac} = e \underline{r} \times \underline{A} . \quad -(15)$$

Here $\underline{A}$ is the vector potential of the AB vacuum. In this theory the angular momentum $\underline{L}_{vac}$ is considered to be a classical function. Eq (14) is quantized as:

$$H_{LS} \phi = - \frac{e^2 m_s \hbar L_Z (vac)}{2\pi \epsilon_0 m^2 c^2 (1+\gamma)^2 r^3} \phi \quad -(16)$$

where

$$m_s = -S, \ldots, S \quad -(17)$$

$$S = 1/2 \quad -(18)$$

and

For the $2P_{1/2}$ state of H:

$$m_s = -\frac{1}{2} \quad -(19)$$

so:

$$E_{LS} = \frac{e^2 \hbar}{4(1+\gamma)^2 \epsilon_0 m^2 c^2} \left\langle \frac{1}{r^3} \right\rangle L_Z (vac) \quad -(20)$$

where the expectation value of $1/r^3$ in H is well known to be:

4)
$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a_0^3 n^3 L\left(L+\frac{1}{2}\right)(L+1)} \quad -(21)$$

The fine structure constant is:
$$\alpha = \frac{e^2}{4\pi \hbar c \epsilon_0} \quad -(22) \quad -(23)$$

So
$$\boxed{E_{LS} = \frac{\alpha\, c}{(1+\gamma)^2}\left(\frac{\hbar}{mc}\right)^2 \left\langle \frac{1}{r^3} \right\rangle L_Z(vac)}$$

The Lamb shift is:
$$f = 10^9\ Hz \quad -(24)$$

and
$$E_{LS} = \hbar\, 2\pi f \quad -(25)$$

In the limit:
$$\gamma \to 1 \quad -(26)$$

it follows that
$$2\pi f \hbar = \frac{\alpha c}{4}\left(\frac{\hbar}{mc}\right)^2 \frac{L_Z(vac)}{a_0^3 n^3 L\left(L+\frac{1}{2}\right)(L+1)} \quad -(27)$$

Now denote
$$\lambda_c = \frac{\hbar}{mc} \quad -(28)$$

so the vacuum angular momentum is given by the following equation:

5)

$$\boxed{L_z(vac) = \frac{8\pi f \hbar a_0^3 L(L+1/2)(L+1)}{\alpha c \lambda_c^2}} \quad -(29)$$

Units Check

$$RHS = \frac{s^{-1} J s m^3}{m s^{-1} m^2} = Js$$

Here:

$$\alpha = 0.007297351, \; f = 10^9 Hz \; a_0 = 5.2918 \times 10^{-11} m,$$

$$n = 2, L = 1, c = 2.997925 \times 10^8 \, ms^{-1}, \text{ and} \quad -(30)$$

$$\lambda_c = \frac{2.426309}{2\pi} \times 10^{-12} = 3.862 \times 10^{-13} m$$

So:

$$L_z(vac) = \frac{8 \times 3.1415927 \times 10^9 \times 1.0546 \times 10^{-34} \times 15 \times 5.2918^3 \times 10^{-33}}{0.007297 \times 2.998 \times 10^8 \times 3.862^2 \times 10^{-26}} \quad -(31)$$

$$= \frac{120 \times 3.1415927 \times 1.0546}{0.007297 \times 2.998 \times 3.862^2} \times 10^{-40}$$

$$= 1.218 \times 10^{-37} \, kg \, m s^{-1}$$

To order of magnitude:

$$L_z = erA = eB(\text{eff}) \quad -(32)$$

where:

$$e = 1.6022 \times 10^{-19}\ C \quad -(33)$$

so:

$$B(\text{eff}) = \frac{1.218 \times 10^{-37}}{1.6022 \times 10^{-19}} \quad -(34)$$

$$= 7.602 \times 10^{-19}\ \text{tesla}$$

If it is assumed that:

$$r \sim a_0 \quad -(35)$$

in eq. (32) then the vacuum vector potential is

$$A = \frac{L_Z}{e a_0} \quad -(36)$$

$$= \frac{1.218 \times 10^{-37}}{5.2918 \times 10^{-11} \times 1.6022 \times 10^{-19}}$$

$$= 1.437 \times 10^{-8}\ \text{tesla metres}$$

So:

$$A = 1.437 \times 10^{-8}\ \text{tesla metres}$$
$$B = 7.602 \times 10^{-19}\ \text{tesla} \quad -(37)$$

if

$$r \sim a_0 \quad -(38) \qquad -(39)$$

and

$$L_Z(\text{vac}) = 1.218 \times 10^{-37}\ \text{kg m s}^{-1}$$

In this view the Lamb shift is caused by a vacuum vector potential of the above magnitude. The

7) angular momentum in the H atom is defined for a given hydrogenic orbital by:

$$L^2 \phi = L(L+1)\hbar^2 \phi \quad -(40)$$

and

$$L_Z \phi = m_L \hbar \phi, \quad -(41)$$

So in the H atom the angular momentum is of the order of the reduced Planck constant:

$$\hbar = 1.0546 \times 10^{-34} \text{ Js.} \quad -(42)$$

The angular momentum imparted by the AB vacuum is:

$$L_Z(\text{vac}) = 1.218 \times 10^{-37} \text{ Js} \quad -(43)$$

so the theory makes sense.

From eq. (29) $L_Z(\text{vac})$ is non-zero if and only if L is non-zero. So it increases the energy of the $^2P_{1/2}$ state but leaves the $^2S_{1/2}$ state unchanged.