332(5): The Anomalous Zeeman Effect from the Rigorously Correct Dirac Equation

In the O(3) basis it has been shown in previous notes and papers that the rigorously correct hamiltonian of special relativity is:

$$H_0 = \left(\frac{\gamma^2}{1+\gamma}\right)\frac{p_0^2}{m} + U \quad -(1)$$

where $\underline{p_0}$ is the non relativistic momentum:

$$\underline{p_0} = m\underline{v_0} \quad -(2)$$

and γ the Lorentz factor:

$$\gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(3)$$

$$-(4)$$

In the SU(2) basis:

$$H_0 = \frac{1}{m}\,\underline{\sigma}\cdot\underline{p_0}\left(\frac{\gamma^2}{1+\gamma}\right)\underline{\sigma}\cdot\underline{p_0} + U \quad -(5)$$

In the previous note it was shown that:

$$\frac{\gamma^2}{1+\gamma} \sim \frac{1}{2}\left(1 - \frac{U}{2mc^2} + \frac{1}{mc^2}\left(\frac{H_0}{2} + \frac{p_0^2}{2m}\right)\right)$$

In the Dirac approximation:

$$\frac{\gamma^2}{1+\gamma} \sim \frac{1}{2}\left(1 - \frac{U}{2mc^2}\right). \quad -(6)$$

So there are two terms missing in the Dirac approximation. The latter was shown to produce:

$$H_0 =? \, 0. \quad -(7)$$

2) Despite its restrictive nature, the Dirac approximation has been used for ninety years uncritically! Therefore, many useful spectra have been missed entirely!

In order to illustrate this situation we first consider the Zeeman effect in the presence of spin - the anomalous Zeeman effect.

In the Dirac approximation:

$$H_0 = \frac{p_0^2}{2m} - \frac{1}{4m^2c^2}\,\underline{\sigma}\cdot\underline{p}_0\, U\, \underline{\sigma}\cdot\underline{p}_0 + U \quad -(8)$$

The effect of a magnetic field is developed in the minimal prescription:

$$\underline{p}_0 \rightarrow \underline{p}_0 - e\underline{A} \quad -(9)$$

The anomalous Zeeman effect is produced by the first term on the right hand side of eq. (8), which in the presence of a magnetic field becomes:

$$H_0 = \frac{1}{2m}\left(\underline{p}_0 - e\underline{A}\right)\cdot\left(\underline{p}_0 - e\underline{A}\right) \quad -(10)$$

In the SU(2) basis:

$$H_0 = \frac{1}{2m}\,\underline{\sigma}\cdot\left(\underline{p}_0 - e\underline{A}\right)\underline{\sigma}\cdot\left(\underline{p}_0 - e\underline{A}\right), \quad -(11)$$

$$H_0\phi = \left(-\frac{e}{2m}\,\underline{L}_0\cdot\underline{B} - \frac{e\hbar}{2m}\,\underline{\sigma}\cdot\underline{B}\right)\phi \quad -(12)$$

3) The first term on the right hand side of eq. (12) gives the normal Zeeman effect using:

$$L_0 \phi = m_L \hbar \phi \quad -(13)$$

with

$$\Delta m_L = 0, \pm 1 \quad -(14)$$

The spin term originates in:

$$H_0 = -\frac{e}{2m} \underline{\sigma} \cdot \underline{p_0} \underline{\sigma} \cdot \underline{A} + \dots \quad -(15)$$

Quantize using:

$$-i\hbar \underline{\nabla} \phi = \underline{p_0} \phi \quad -(16)$$

to find:

$$H_0 \phi = \left(\frac{ie\hbar}{2m} \underline{\sigma} \cdot \underline{\nabla} \underline{\sigma} \cdot \underline{A} + \dots \right) \phi \quad (17)$$

where

$$\underline{\sigma} \cdot \underline{\nabla} \underline{\sigma} \cdot \underline{A} = \underline{\nabla} \cdot \underline{A} + i \underline{\sigma} \cdot \underline{\nabla} \times \underline{A} \quad -(18)$$

with

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(19)$$

so:

$$H_0 \phi = -\frac{e\hbar}{2m} \underline{\sigma} \cdot \underline{B} \quad -(20)$$

QED. using

$$\underline{\hat{S}} = \frac{\hbar}{2} \underline{\hat{\sigma}} \quad -(21)$$

Eq. (12) becomes:

$$H_0 \phi = \left(\underline{\hat{L}_0} + 2 \underline{\hat{S}} \right) \cdot \underline{B} \phi \quad -(22)$$

4) Align $\underline{B}$ in the Z axis:

$$H_0\phi = -\frac{e}{2m}\left(\hat{L}_{0Z} + 2\hat{S}_Z\right)B_Z\phi \quad -(23)$$

where

$$\hat{L}_{0Z}\phi = \hbar m_L \phi \quad -(24)$$

$$\hat{S}_Z\phi = \hbar m_S \phi \quad -(25)$$

and

$$m_L = -L, \ldots L \quad -(26)$$

$$m_S = -S, \ldots S = \pm\frac{1}{2} \quad -(27)$$

The anomalous Zeeman effect was explained by Landé, of the Sommerfeld group. The method used was to develop the hamiltonian (22) using vector algebra as follows:

$$\underline{L}_0 \cdot \underline{B} = \underline{L}_0 \cdot \underline{k}\,\underline{k} \cdot \underline{B} \quad -(28)$$

$$\underline{S} \cdot \underline{B} = \underline{S} \cdot \underline{k}\,\underline{k} \cdot \underline{B} \quad -(29)$$

where

$$\underline{k} = \frac{\underline{J}}{J} \quad -(30)$$

By restricting attention to Z components:

$$\underline{L}_0 \cdot \underline{B} = L_{0Z}B_Z = L_{0Z}J_Z J_Z B_Z / J_Z^2 \quad -(31)$$

QED. Only the Z components are of interest in QM.

So:

$$\underline{L}_0 \cdot \underline{B} = \frac{1}{J^2}\underline{L}_0 \cdot \underline{J}\,\underline{J} \cdot \underline{B} \quad -(32)$$

$$\underline{S} \cdot \underline{B} = \frac{1}{J^2}\underline{S} \cdot \underline{J}\,\underline{J} \cdot \underline{B} \quad -(33)$$

5) Now use:

$$2\underline{L}_0 \cdot \underline{J} = J^2 + L_0^2 - |\underline{J} - \underline{L}_0|^2 = J^2 + L_0^2 - S^2 \quad -(34)$$

and

$$2\,\underline{S} \cdot \underline{J} = J^2 + S^2 - |\underline{J} - \underline{S}|^2 = J^2 + S^2 - L_0^2 \quad -(35)$$

So:

$$(\underline{L}_0 + 2\underline{S}) \cdot \underline{B} = \left(\frac{1}{2} \frac{(J^2 + L_0^2 - S^2) + J^2 + S^2 - L_0^2}{J^2} \right) \underline{J} \cdot \underline{B}$$

$$= \frac{1}{2} \left(1 + \frac{J^2 + S^2 - L_0^2}{J^2} \right) \underline{J} \cdot \underline{B} \quad -(36)$$

$$:= g_L \, \underline{J} \cdot \underline{B}$$

where

$$g_L = \frac{1}{2} \left(1 + \frac{J^2 + S^2 - L_0^2}{J^2} \right) \quad -(37)$$

Considering expectation values:

$$\langle H_0 \rangle = -\frac{e}{2m} \left(\langle \underline{L}_0 \rangle + 2 \langle \underline{S} \rangle \right) \cdot \underline{B} \quad -(38)$$

where

$$L_0^2 \psi = \hbar^2 L(L+1) \psi \quad -(39)$$

$$S^2 \psi = \hbar^2 S(S+1) \psi$$

$$J^2 \psi = \hbar^2 J(J+1) \psi$$

so

$$\langle H_0 \rangle = -\frac{e}{2m} g_L \langle \underline{J} \rangle \cdot \underline{B} \quad -(40)$$

where:

6)
$$g_L = \frac{1}{2}\left(1 + \frac{\langle J^2\rangle + \langle S^2\rangle - \langle L_0^2\rangle}{\langle J^2\rangle}\right) \quad -(40)$$

$$= \frac{1}{2}\left(1 + \frac{J(J+1) + S(S+1) - L(L+1)}{J(J+1)}\right)$$

is <u>the Landé factor.</u>

Therefore the energy levels of the anomalous Zeeman effect are:

$$E = \langle H_0 \rangle = -\frac{e}{2m} g_L \langle J_Z \rangle B_Z \quad -(41)$$

where
$$\langle J_Z \rangle = \hbar m_J \quad -(42)$$

with
$$\Delta m_J = 0, \pm 1 \quad -(43)$$

and
$$\Delta J = 0, \pm 1 \quad -(44)$$

$$J = 0 \not\rightarrow J = 0 \quad -(45)$$

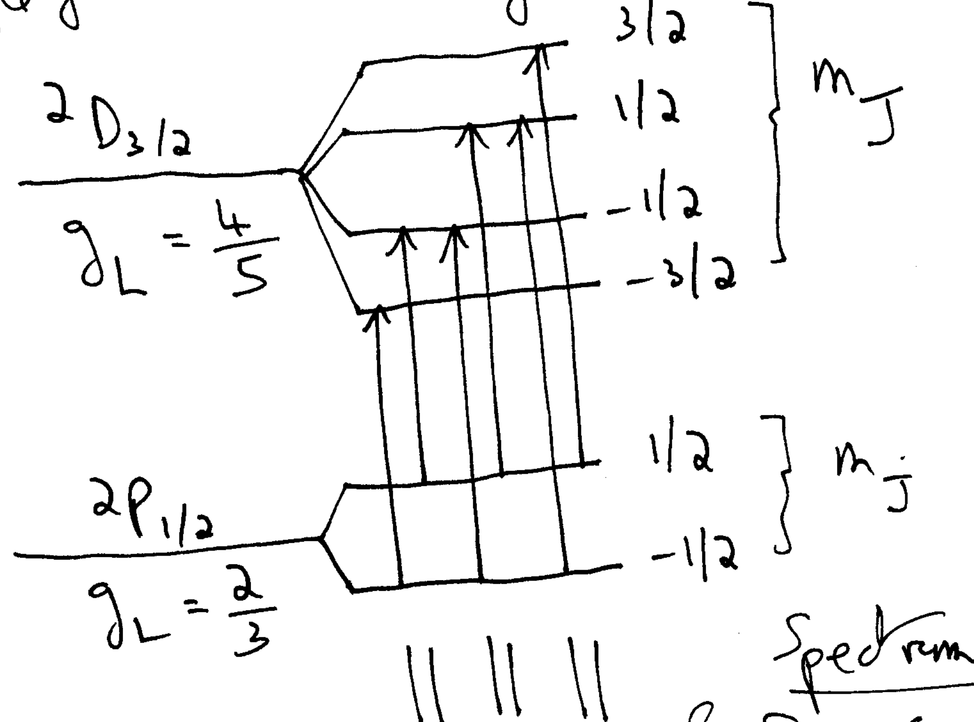
The normal Zeeman effect is:

$$E = -\frac{e\hbar}{2m} m_L B_Z \quad -(46)$$

with
$$\Delta m_L = 0, \pm 1 \quad -(47)$$

giving the well known splitting into three lines.

7) In the anomalous Zeeman effect each of these three lines is split according to Eq. (41). The latter described the experimental data known in Landé's time. For example for the Hα line of atomic hydrogen:



The D and P terms have different Landé factors so the three lines of the ordinary or normal Zeeman effect are split into three doublets, as observed experimentally.

However, the correct eq. (5) gives:

$$E = \langle H_0 \rangle - \frac{e\hbar}{2m} g_L m_J B_Z + \frac{1}{mc^2} \left\langle \frac{H_0}{2} + \frac{p_0^2}{2m} \right\rangle \left\langle \frac{p_0^2}{2m} \right\rangle \quad -(48)$$

As in note 332(4):

$$\frac{1}{mc^2} \left\langle \frac{H_0}{2} + \frac{p_0^2}{2m} \right\rangle = \frac{1}{4} \left(\frac{\lambda_c}{a_0} \right) \frac{\alpha}{n^2} \quad -(49)$$

8) where λ_c is the Compton wavelength of the electron, α is the fine structure constant, a_0 is the Bohr radius, and n the principal quantum number of atomic H.

So the anomalous Zeeman effect becomes:

$$\boxed{E = \langle H_0 \rangle = -\frac{e\hbar}{2m} g_L m_J B_Z + \frac{1}{4}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2}\left\langle \frac{p_0^2}{2m} \right\rangle} \quad -(50)$$

The energy levels E are added to the energy levels of the H atom:

$$E_H = -mc^2\left(\frac{\alpha}{n^2}\right) = -\frac{me^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(51)$$

so the complete effect gives the energy levels:

$$E_1 = -mc^2\left(\frac{\alpha}{n^2}\right) - \frac{e\hbar}{2m} g_L m_J B_Z + \frac{1}{4}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2}\left\langle \frac{p_0^2}{2m} \right\rangle \quad -(52)$$

$$= -mc^2\left(\frac{\alpha}{n^2}\right)\left(1 - \frac{1}{4}\left(\frac{\lambda_c}{a_0}\right)\left(\frac{\alpha}{n^2}\right)\right) - \frac{e\hbar}{2m} g_L m_J B_Z$$

where we have used:

$$\left\langle \frac{p_0^2}{2m} \right\rangle = mc^2\left(\frac{\alpha}{n^2}\right) \quad -(53)$$