322: The Correct Quantization of the Einstein Energy Equation.

As in previous notes and notation express the Einstein energy equation as:

$$H_0 = H - mc^2 = \frac{p^2}{(1+\gamma)m} + U \quad -(1)$$

where

$$p^2 = \gamma^2 p_0^2 \quad -(2)$$

so

$$\boxed{H_0 = \left(\frac{\gamma^2}{1+\gamma}\right)\frac{p_0^2}{m} + U} \quad -(3)$$

In the limit:

$$\gamma \to 1 \quad -(4)$$

this reduces to the Schroedinger equation:

$$H_0 = \frac{p_0^2}{2m} + U \quad -(4)$$

using

$$-i\hbar \underline{\nabla} \phi = \underline{p_0} \phi \quad -(5)$$

Therefore the wave functions to be used in eq. (4) are the non relativistic wave functions.

In the SU(2) basis eq. (3) can be expressed in two ways:

2)
$$H_0 = \frac{1}{m} \underline{\sigma} \cdot \underline{p}_0 \frac{\gamma^2}{1+\gamma} \underline{\sigma} \cdot \underline{p}_0 + U \quad -(6)$$

or
$$H_0 = \frac{1}{m} \underline{\sigma} \cdot \underline{p}_0 \, \underline{\sigma} \cdot \underline{p}_0 \frac{\gamma^2}{1+\gamma} + U \quad -(7)$$
$$= \left(\frac{\gamma^2}{1+\gamma}\right) \frac{p_0^2}{m} + U$$

After quantization eqs. (6) and (7) give different results. Eq. (6) gives a relativistically corrected spin orbit coupling but eq. (7) does not.

The correct Lorentz factor is:
$$\gamma = \left(1 - \frac{p_0^2}{m^2 c^2}\right)^{-1/2} \quad -(8)$$

Now use:
$$\frac{\gamma^2}{1+\gamma} = \left(\frac{1}{\gamma^2} + \frac{1}{\gamma}\right)^{-1} \quad -(9)$$
$$= \left(1 - \frac{p_0^2}{m^2 c^2} + \left(1 - \frac{p_0^2}{m^2 c^2}\right)^{1/2}\right)^{-1}$$

In the limit:
$$v \ll c \quad -(10)$$
$$\frac{\gamma^2}{1+\gamma} = \frac{1}{2 - \frac{3p_0^2}{m^2 c^2}} = \frac{1}{2}\left(1 - \frac{3p_0^2}{4m^2 c^2}\right)^{-1} \quad -(11)$$

3) In the limit (10):

$$\frac{\gamma^2}{1+\gamma} \sim \frac{1}{2}\left(1+\frac{3p_0^2}{4m^2c^2}\right) \quad -(12)$$

where

$$p_0^2 = 2m(H_0 - U) \quad -(13)$$

Therefore:

$$\frac{\gamma^2}{1+\gamma} = \frac{1}{2}\left(1 + \frac{3(H_0 - U)}{2mc^2}\right) \quad -(14)$$

$$= \frac{1}{2}\left(1 - \frac{U}{2mc^2} + \frac{1}{2mc^2}(3H_0 - 2U)\right)$$

$$= \frac{1}{2}\left(1 - \frac{U}{2mc^2} + \frac{H_0}{2mc^2} + \frac{1}{mc^2}(H_0 - U)\right)$$

$$= \frac{1}{2}\left(1 - \frac{U}{2mc^2} + \frac{1}{mc^2}\left(\frac{H_0}{2} + \frac{p_0^2}{2m}\right)\right)$$

As in Note 331(5):

$$\left\langle \frac{p_0^2}{2m^2c^2} \right\rangle = \frac{1}{2}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2} \quad -(15)$$

and

$$\left\langle \frac{H_0}{2mc^2} \right\rangle = -\frac{1}{4}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2} \quad -(16)$$

Therefore:

$$\left\langle \frac{1}{mc^2}\left(\frac{H_0}{2}+\frac{p_0^2}{2m}\right)\right\rangle = \frac{1}{4}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \quad -(17)$$

From eqs. (6) and (14):

$$H_0 = \frac{1}{2m}\underline{\sigma}\cdot\underline{p_0}\left(1-\frac{U}{2mc^2}+\frac{1}{mc^2}\left(\frac{H_0}{2}+\frac{p_0^2}{2m}\right)\right)\underline{\sigma}\cdot\underline{p_0} + U \quad -(18)$$

Now quantize using eq. (5):

$$H_0\phi = \frac{1}{2m}\left(\underline{\sigma}\cdot\underline{p_0}\left(1-\frac{U}{2mc^2}\right)\underline{\sigma}\cdot\underline{p_0}\,\phi + \underline{\sigma}\cdot\underline{p_0}\,\frac{1}{mc^2}\left(\frac{H_0}{2}+\frac{p_0^2}{2m}\right)\underline{\sigma}\cdot\underline{p_0}\,\phi\right.$$

$$= \frac{p_0^2}{2m}\left(1+\frac{1}{mc^2}\left(\frac{H_0}{2}+\frac{p_0^2}{2m}\right)\right)\phi \quad -(19)$$

$$-\frac{1}{4m^2c^2}\,\underline{\sigma}\cdot\underline{p_0}\,U\,\underline{\sigma}\cdot\underline{p_0}\,\phi$$

The first term in eq. (19) gives the hyperfine structure in the Zeeman effect, the second term gives spin orbit coupling.