Note 332(4): Evaluation of The Hamiltonian of Note 332(2).

This hamiltonian is:

$$H_0 \phi = \frac{p_0^2}{2m}\left(1 + \frac{1}{mc^2}\left(\frac{H_0}{2} + \frac{p_0^2}{2m}\right)\right)\phi \quad -(1)$$

$$- \frac{1}{4m^2c^2}\, \underline{\sigma} \cdot \underline{p_0}\, U\, \underline{\sigma} \cdot \underline{p_0}\, \phi$$

which is quantized with:

$$-\hbar^2 \nabla^2 \phi = p_0^2 \phi. \quad -(2)$$

Therefore:

$$H_0 \phi = -\left(1 - \frac{H_0}{2mc^2}\right)\frac{\hbar^2 \nabla^2}{2m}\phi - \frac{\hbar^2}{mc^2}\frac{p_0^2}{2m}\frac{\nabla^2 \phi}{2m}$$

$$- \frac{1}{4m^2c^2}\, \underline{\sigma} \cdot \underline{p_0}\, U\, \underline{\sigma} \cdot \underline{p_0}\, \phi$$

$$= -\left(1 - \frac{H_0}{2mc^2}\right)\frac{\hbar^2 \nabla^2}{2m}\phi - \frac{1}{4m^2c^2}\, \underline{\sigma} \cdot p_0\, U\, \underline{\sigma} \cdot p_0\, \phi$$

$$- \frac{\hbar^2}{mc^2}\frac{p_0^2}{2m}\frac{\nabla^2 \phi}{2m} \quad -(3)$$

In this equation:

$$\left\langle \frac{H_0}{2mc^2} \right\rangle = -\frac{1}{4}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2} \quad -(4)$$

$$\left\langle \frac{p_0^2}{2m^2c^2} \right\rangle = \frac{1}{2}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2} \quad -(5)$$

and

$$\left\langle -\frac{\hbar^2\nabla^2}{2m}\right\rangle = \frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} = mc^2\left(\frac{\alpha}{n}\right)^2 \quad -(6)$$

Therefore:

$$H_0 = \langle \psi^* \hat{H}_0 \psi \rangle = \left(1 - \left\langle \frac{H_0}{2mc^2}\right\rangle\right)\left\langle -\frac{\hbar^2\nabla^2}{2m}\right\rangle$$
$$- \left\langle \frac{1}{4m^2c^2} \underline{\sigma}\cdot\underline{p_0}\, U\, \underline{\sigma}\cdot\underline{p_0}\right\rangle + \left\langle \frac{p_0^2}{2m^2c^2}\right\rangle\left\langle -\frac{\hbar^2\nabla^2}{2m}\right\rangle$$
$$-(7)$$

In this expression:

$$- \left\langle \frac{1}{4m^2c^2} \underline{\sigma}\cdot\underline{p_0}\, U\, \underline{\sigma}\cdot\underline{p_0}\right\rangle \quad -(8)$$
$$= \left\langle \frac{e^2}{16\pi\epsilon_0 m^2c^2r^3}\left(J(J+1) - L(L+1) - S(S+1)\right)\right\rangle$$
$$= \frac{e^2}{16\pi\epsilon_0 m^2c^2}\left(\frac{J(J+1) - L(L+1) - S(S+1)}{a_0^3 n^3 L\left(L+\frac{1}{2}\right)(L+1)}\right)$$

Therefore:

$$H_0 = \left(1 + \frac{1}{4}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2}\right) mc^2 \frac{\alpha^2}{n^2} + \frac{1}{2}\left(\frac{\lambda_c}{a_0}\right)\frac{\alpha}{n^2}\frac{mc^2\alpha^2}{n^2}$$
$$+ \frac{e^2}{16\pi\epsilon_0 m^2c^2}\left(\frac{J(J+1) - L(L+1) - S(S+1)}{a_0^3 n^3 L\left(L+\frac{1}{2}\right)(L+1)}\right)$$
$$-(9)$$

In the Dirac approximation:

$$H_0 = mc^2\left(\frac{\alpha}{n}\right)^2 - \frac{e^2}{16\pi\epsilon_0 m^2c^2}\left(\frac{J(J+1)-L(L+1)-S(S+1)}{a_0^3 n^3 L\left(L+\frac{1}{2}\right)(L+1)}\right) \quad -(10)$$

On the classical level the Dirac approximation results in

$$H_0 = 0 \quad -(11)$$

as shown in Note 332(1), so results in the clearly incorrect:

$$mc^2\left(\frac{\alpha}{n}\right)^2 =? \frac{e^2}{16\pi\epsilon_0 m^2c^2}\left(\frac{J(J+1)-L(L+1)-S(S+1)}{a_0^3 n^3 L\left(L+\frac{1}{2}\right)(L+1)}\right) \quad -(12)$$

The spectrum from the correct result (9) may now be evaluated using transition rules:

$$\Delta L = 1, \quad \Delta J = 0, \pm 1 \quad -(13)$$

and

$$J = 0 \not\rightarrow J = 0 \quad -(14)$$

Finally the entire exercise may be repeated in the presence of a magnetic field, as in the next note

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