Note 326(1): Calculation of the Velocity Curve

Start by noting that the relativistic momentum is:

$$L = \gamma m r^2 \dot{\theta} = \gamma L_0 \qquad (1)$$

So

$$v_N^2 = \frac{L^2}{\gamma^2 m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \qquad (2)$$

$$= \frac{L_0^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)$$

$$= v_N^2$$

where:

$$\gamma = \left(1 - \left(\frac{v_N}{c}\right)^2\right)^{-1/2} \qquad (3)$$

So:

$$v_N^2 = \frac{L^2}{m^2}\left(\left(1 - \left(\frac{v_N}{c}\right)^2\right)\right)\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \qquad (4)$$

and

$$v_N^2\left(1 + \frac{L^2}{c^2 m^2}\right) = \frac{L^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)$$

i.e.

$$\boxed{v_N^2 = \frac{v^2}{1 + \frac{v^2}{c^2}}} \qquad (5)$$

It follows that:

$$v^2 = \left(1 + \frac{v^2}{c^2}\right) v_N^2 \quad -(6)$$

i.e.

$$v^2 \left(1 - \frac{v_N^2}{c^2}\right) = v_N^2 \quad -(7)$$

or

$$\boxed{v = \gamma v_N} \quad -(8)$$

where

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(9)$$

Eq. (8) defines the relativistic momentum:

$$\underline{p} = \gamma m \underline{v} \quad -(10)$$

Q.E.D.

This result is self consistent because the relativistic lagrangian is defined to give the relativistic momentum.

From eq. (8):

$$v^2 = \frac{\gamma^2 L_0^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad -(11)$$

$$= \frac{L^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad -(12)$$