326(8): General Quantization Scheme

Consider the Einstein energy equation:

$$E^2 = p^2c^2 + m^2c^4 \quad -(1)$$

where

$$E = H - U \quad -(2)$$

Here E is the total relativistic energy:

$$E = \gamma mc^2, \quad -(3)$$

p is the relativistic momentum:

$$\underline{p} = \gamma m \underline{v} \quad -(4)$$

and U the potential energy. The lagrangian is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(5)$$

so

$$E = H + \mathcal{L} + \frac{mc^2}{\gamma} \quad -(6)$$

Eq. (1) can be expressed either as:

1)
$$E^2 - m^2c^4 = p^2c^2 \quad -(7)$$

or:

2)
$$E^2 - p^2c^2 = m^2c^4 \quad -(8)$$

Eq. (7) leads to:

$$E - mc^2 = \frac{p^2c^2}{E + mc^2} \quad -(9)$$

 and eq. (8) leads to:

$$E - pc = \frac{m^2 c^4}{E + pc} \quad -(10)$$

Both eqs. (9) and (10) lead to a large variety of new effects.

To date in the UFT series only eq. (9) has been developed, using the Dirac approximation. Both eqs (9) and (10) can be developed in terms of the lagrangian, so a number of new effects can be inferred from the Euler Lagrange equations. In these developments E is always defined by:

$$\boxed{E = H - U = H + \mathcal{L} + \frac{mc^2}{\gamma} = \gamma m c^2} \quad -(11)$$

and in general:

$$p^2 = \gamma^2 m^2 v_0^2 \quad -(12)$$

where v_0 is the observer frame velocity of a particle.

Eq. (1) can be written as:

$$p^\mu p_\mu = m^2 c^2 \quad -(13)$$

 where:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(14)$$

and

$$p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(15)$$

and Schroedinger quantization can be applied as follows:

$$p^\mu = i\hbar\partial^\mu \quad -(16)$$

where

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad -(17)$$

and

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right) \quad -(18)$$

Therefore the relativistic momentum is quantized as:

$$\underline{p}\phi = -i\hbar\underline{\nabla}\phi \quad -(19)$$

and the relativistic energy by:

$$E\phi = i\hbar\frac{\partial\phi}{\partial t} \quad -(20)$$

Scheme One

This is the quantization of eq. (9). The rigorous quantization is carried out with:

$$E = \gamma mc^2 \quad -(21)$$

so

$$mc^2(\gamma - 1) = \frac{p^2c^2}{mc^2(\gamma+1)} \quad -(22)$$

So
$$p^2c^2\phi = (\gamma^2 - 1)m^2c^4\phi \quad -(23)$$
in which
$$\gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(24)$$
where
$$p^2 = \gamma^2 m^2 v_0^2 \quad -(25)$$
Therefore:
$$\gamma^2 = 1 + \left(\frac{p}{mc}\right)^2 \quad -(26)$$
and
$$\boxed{\frac{p^2}{2m}\phi = \frac{mc^2}{2}(\gamma^2 - 1)\phi} \quad -(27)$$
In the non relativistic limit:
$$\frac{p^2}{2m} = mc^2\left(\left(1 - \frac{v_0^2}{c^2}\right)^{-1} - 1\right) \quad -(28)$$
$$\xrightarrow[v_0 \ll c]{} \frac{1}{2}mv_0^2$$
QED. Eq. (27) is equivalent to:
$$\frac{p^2}{2m}\phi = \frac{mc^2}{2}\left(\frac{E^2}{m^2c^4} - 1\right)\phi \quad -(29)$$
(which is a quantized version of eq (1), QED.)

5) Eq. (29) can be developed with:

$$E = H - U \quad -(30)$$

so:

$$\frac{p^2}{2m}\phi = \frac{mc^2}{2}\left(\frac{(H-U)^2}{m^2c^4} - 1\right)\phi \quad -(31)$$

$$= \left(H - U - mc^2\right)\left(\frac{1}{2}\left(1 + \frac{H-U}{mc^2}\right)\right)\phi$$

so

$$\left(\frac{p^2}{m}\left(1 + \frac{H-U}{mc^2}\right)^{-1} + U\right)\phi = \left(H - mc^2\right)\phi \quad -(32)$$

i.e

$$\left(\frac{p^2}{m}\left(1 + \frac{E}{mc^2}\right)^{-1} + U\right)\phi = \left(H - mc^2\right)\phi \quad -(33)$$

or

$$\left(\frac{p^2}{m}\left(1+\gamma\right)^{-1} + U\right)\phi = H_1\phi \quad -(34)$$

where

$$H_1 := H - mc^2 \quad -(35)$$

Eq. (34) is the Schroedinger equation:

$$\left(\frac{p^2}{2m} + U\right)\phi = H_1\phi \quad -(36)$$

in the limit

$$\gamma \to 1 \quad -(37)$$

6) Eq. (34) is the relativistic Schroedinger equation in rigorous format, QED. Therefore:

$$\frac{p^2}{2m} \longrightarrow \frac{p^2}{m(1+\gamma)} \quad -(38)$$

where:

$$\gamma^2 = 1 + \left(\frac{p}{mc}\right)^2 = \frac{E^2}{m^2c^4} \quad -(39)$$

so

$$\boxed{H_1 \psi = \left(\frac{p^2}{m\left(1+\frac{E}{mc^2}\right)} + U\right)\psi} \quad -(40)$$

where

$$E = H - U \quad -(41)$$

Eq. (40) is particularly useful as discussed in previous UFT papers. Here it has been derived rigorously. The approximation used by Dirac is:

$$H = E + U \sim mc^2 \quad -(42)$$

so

$$\frac{U}{mc^2} \ll 1 \quad -(43)$$

and

$$\gamma \rightarrow 1 \quad -(44)$$

However the rigorous result is:

1)

$$\boxed{\begin{aligned} H_1 \phi &:= \left(H - mc^2\right)\phi \\ &= \left(\frac{p^2}{m\left(1 + \frac{H}{mc^2} - \frac{U}{mc^2}\right)} + U\right)\phi \end{aligned}} \quad -(45)$$

where H is a constant of motion:

$$\frac{dH}{dt} = 0 \quad -(46)$$

The above quantization is equivalent to:

$$\boxed{\frac{p^2}{2m} = \left(H - U - mc^2\right)\left(\frac{E + mc^2}{2mc^2}\right)} \quad -(47)$$

This reduces to the Schroedinger equation:

$$\frac{p^2}{2m}\phi = \left(H - U - mc^2\right)\phi \quad -(48)$$

as

$$E \longrightarrow mc^2, \; \gamma \to 1 \quad -(49)$$

Using:

$$E = \gamma mc^2 \quad -(50)$$

it is found that:

$$H - U - mc^2 = \left(\frac{2}{1+\gamma}\right)\frac{p^2}{2m} = \frac{p^2}{(1+\gamma)m} \quad -(51)$$

So:

$$H_1 = H - mc^2 = \frac{p^2}{(1+\gamma)m} + U \quad -(52)$$

where

$$\gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(53)$$

The Schroedinger equation is therefore modified to:

$$\left(\left(\frac{2}{1+\gamma}\right)\frac{p^2}{2m} + U\right)\phi = \left(H - mc^2\right)\phi \quad -(54)$$

where:

$$p^2 = -\hbar^2 \nabla^2 \quad -(55)$$

Therefore:

$$\left(\frac{p^2}{2m} + \frac{1}{2}(1+\gamma)U\right)\phi = \frac{1}{2}(1+\gamma)\left(H - mc^2\right)\phi \quad -(56)$$

Now use:

$$\frac{1}{2}(1+\gamma)U = U + \frac{1}{2}(\gamma - 1)U \quad -(57)$$

$$-(58)$$

and

$$\frac{1}{2}(1+\gamma)\left(H - mc^2\right) = \left(H - mc^2\right) + \frac{1}{2}(\gamma - 1)\left(H - mc^2\right)$$

$$-(59)$$

Eq. (56) is therefore:

$$\left(\frac{p^2}{2m} + U\right)\phi = \left(H - mc^2\right)\phi + \left[\frac{1}{2}(1-\gamma)U + \frac{1}{2}(\gamma - 1)\left(H - mc^2\right)\right]\phi$$

i.e.

$$\boxed{\left(\frac{p^2}{2m}+U\right)\phi = \left(H-mc^2\right)\phi + \frac{1}{2}(\gamma-1)\left[H-mc^2-U\right]\phi} \quad -(60)$$

This reduces to the Schroedinger equation when

$$\gamma \to 1 \quad -(61)$$

Q.E.D.

In the usual notation of the Schroedinger equation:

$$\boxed{E_{total} = H - mc^2} \quad -(62)$$

where E_{total} is the total energy. So:

$$\left(\frac{p^2}{2m}+U\right)\phi = E_{total}\phi + \frac{1}{2}(\gamma-1)\left(E_{total}-U\right)\phi \quad -(63)$$

The energy levels of the H atom are given by:

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(64)$$

so the energy levels are shifted by:

$$\boxed{E_{total} \to E_{total} + \frac{1}{2}(\gamma-1)\left(E_{total}+\frac{e^2}{4\pi\epsilon_0 r}\right)} \quad -(65)$$

This means that:

$$E_{total} \rightarrow E_{total} + \frac{1}{2}(\gamma - 1)\frac{p^2}{2m} \quad -(66)$$

Finally express this equation in terms of v_0, and v_0 can be measured through the energy shifts of the H atom due to special relativity.

$$\boxed{E_{total} \rightarrow E_{total} + \frac{m^2 v_0^2}{4}\left(\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} - 1\right)} \quad -(67)$$

So the expectation value can be found for each orbital of the H atom:

$$\left\langle \frac{m^2 v_0^2}{4}\left(\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} - 1\right)\right\rangle$$

$$= \int \psi^* \frac{m^2 v_0^2}{4}\left(\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} - 1\right)\psi \, d\tau \quad -(68)$$

$$= \frac{m^2 v_0^2}{4}\left(\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} - 1\right)\int \psi^* \psi \, d\tau$$

The relativistic velocity v of the electron

11) in each orbital of the H atom is:

$$V = V_0 \left(1 - \frac{V_0^2}{c^2}\right)^{-1/2} \quad -(69)$$

<u>Scheme 2</u>

From the equation (9):

$$H - U - mc^2 = \frac{p^2 c^2}{H - U + mc^2} \quad -(70)$$

and in the Dirac approximation:

$$H = \gamma mc^2 + U \rightarrow mc^2 \quad -(71)$$

when $\gamma \rightarrow 1$, and $U << E$ $-(72)$

so $$H - mc^2 = \frac{c^2 p^2}{2mc^2 - U} + U \quad -(73)$$

$$= \frac{p^2}{2m}\left(1 - \frac{U}{2mc^2}\right)^{-1} + U$$

i.e. $$\left(H - mc^2\right)\psi = \left(-\frac{\hbar^2 \nabla^2}{2m}\left(1 + \frac{U}{2mc^2}\right) + U\right)\psi \quad -(74)$$

so: $$\left(-\frac{\hbar^2 \nabla^2}{2m} + U\right)\psi = E_{tot}\psi + \frac{\hbar^2}{4m^2c^2}\nabla^2(U\psi) \quad -(75)$$

and the energy levels of the H atom are shifted by the

Second term on the right hand side.

We have:

$$\underline{\nabla}(U\phi) = U\underline{\nabla}\phi + \phi\underline{\nabla}U \quad -(76)$$

and:

$$\nabla^2(U\phi) = \underline{\nabla}(U\underline{\nabla}\phi + \phi\underline{\nabla}U) \quad -(77)$$

$$= U\nabla^2\phi + 2\underline{\nabla}U\cdot\underline{\nabla}\phi + \phi\nabla^2 U$$

The corrections to the energy levels have expectation values:

$$\langle\Delta E_{tot}\rangle = -\frac{\hbar^2}{4m^2c^2}\left(\int\phi^* U\nabla^2\phi\, d\tau + \int\phi^*(\nabla^2 U)\phi\, d\tau + 2\int\phi^*\underline{\nabla}U\cdot\underline{\nabla}\phi\, d\tau\right) \quad -(78)$$

and these can be evaluated with the assumption that the hydrogenic wave functions can be used.

Quantization Scheme Three

This is based on eq. (10):

$$\boxed{H - U - pc = \frac{m^2c^4}{H - U + pc}} \quad -(79)$$

and will be developed in the next note.