326(6): Relativistic Free Particle Quantization

The basic Lorentz transform is:

$$c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad -(1)$$

where

$$dx'^2 + dy'^2 + dz'^2 = v'^2 dt'^2 \quad -(2)$$

and

$$dx^2 + dy^2 + dz^2 = v^2 dt^2 \quad -(3)$$

In this notation the frame K' moves with respect to the observer frame K at a velocity $\underline{v}$, which is the velocity in the observer frame.

The most basic feature of special relativity is that it introduces the idea of a time t' in the frame K'. This is called the proper time and denoted τ:

$$\tau = t' \quad -(4)$$

The particle moves with frame K' fixed on it, so the velocity $\underline{v}'$ of the particle with respect to K' is zero:

$$\underline{v}' = \underline{0} \quad -(5)$$

Restricting consideration to plane polar coordinates:

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \quad -(6)$$
$$= v^2 dt^2$$

and

2)

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(7)$$

The Lorentz factor γ is defined by:

$$c^2 d\tau^2 = (c^2 - v^2)dt^2 \quad -(8)$$

So

$$\gamma = \frac{dt}{d\tau} = \left(\frac{c^2}{c^2 - v^2}\right)^{1/2} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(9)$$

where v is defined by eq. (7).

Note carefully that in plane polar coordinates:

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \quad -(10)$$

and the classical momentum is:

$$\underline{p}_0 = m\underline{v} \quad -(11)$$

So v in the definition of γ is the classical velocity. In Cartesian coordinates:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dX}{dt}\,\underline{i} + \frac{dY}{dt}\,\underline{j} \quad -(11)$$

The relativistic momentum was introduced by Einstein in 1905 and is:

3)

$$\boxed{\underline{p} = \gamma \underline{p_0} = \gamma m \underline{v}} \quad -(12)$$

This definition is needed through considerations of conservation of linear momentum in special relativity. Therefore the force is:

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(\gamma m \underline{v}) \quad -(13)$$

and this is the Lorentz force equation of ECE 2.

The work done is:

$$W = T = \int \frac{d}{dt}(\gamma m \underline{v}) \cdot \underline{v}\, dt \quad -(14)$$

where T is the relativistic kinetic energy. Therefore:

$$T = W = m \int_0^v v\, d(\gamma v) \quad -(15)$$

Now integrate by parts:

$$T = \gamma m v^2 - m \int_0^v \frac{v\, dv}{(1 - v^2/c^2)^{1/2}}$$

$$= \gamma m v^2 + mc^2 (1 - v^2/c^2)^{1/2} \Big|_0^v$$

$$= \gamma m v^2 + mc^2 (1 - v^2/c^2)^{1/2} - mc^2$$

$$= (\gamma - 1) mc^2 \quad -(16)$$

4) In the non-relativistic limit:

$$v << c \quad - (17)$$

it is found that:

$$T = (\gamma - 1) mc^2 = \left(\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right) mc^2$$

$$\rightarrow \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \ldots - 1 \right) mc^2$$

$$= \frac{1}{2} mv^2 \quad - (18)$$

which is the non-relativistic kinetic energy, QED.

The total energy of a free particle is:

$$E = T + mc^2 \quad - (19)$$

where

$$E_0 = mc^2 \quad - (20)$$

is the rest energy. So:

$$\boxed{E = \gamma mc^2} \quad - (21)$$

The de Broglie / Einstein equations are:

$$E = \gamma mc^2 = \hbar \omega \quad - (22)$$

and

$$\underline{p} = \gamma m \underline{v} = \hbar \underline{\kappa}, \quad - (23)$$

and have been greatly developed in the UFT series.

5) The rest energy and Einstein energy equation are derived from:

$$\underline{p} = \gamma m \underline{v} \quad -(24)$$

so

$$\begin{aligned} c^2 p^2 &= \gamma^2 m^2 v^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) \\ &= \gamma^2 m^2 c^4 - m^2 c^4 \\ &= E^2 - m^2 c^4 \end{aligned} \quad -(25)$$

so

$$\boxed{E^2 = c^2 p^2 + m^2 c^4} \quad -(26)$$

where we have used:

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \quad -(27)$$

Note carefully that the v of the γ factor is the classical velocity. In fact the entire development has used only the classical velocity.

Eq. (26) is:

$$p^\mu p_\mu = m^2 c^2 \quad -(28)$$

where

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(29)$$

$$p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(30)$$

6) Eq. (26) is a powerful equation and leads to the Dirac equation. It can be developed in many ways as in the UFT series. For the free particle, for which it is valid, it can be factorized to give:

$$E^2 - m^2c^4 = (E - mc^2)(E + mc^2) = c^2p^2 \quad -(31)$$

so

$$T = E - mc^2 = \frac{c^2p^2}{E + mc^2} = \frac{c^2p^2}{(\gamma+1)mc^2} \quad -(32)$$

$$= (\gamma - 1)mc^2$$

Therefore:

$$c^2p^2 = (\gamma^2 - 1)m^2c^2 \quad -(33)$$

and

$$\frac{p^2}{2m} = \frac{1}{2}(\gamma^2 - 1)mc^2 \quad -(34)$$

In the limit:

$$v \ll c \quad -(35)$$

$$\frac{p^2}{2m} \rightarrow \frac{1}{2}mv^2$$

$$= \frac{1}{2}\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)mc^2$$

$$= \frac{1}{2}\left(1 + \frac{v^2}{c^2} + \ldots - 1\right)mc^2 \quad -(36)$$

QED

7) Now apply Schrodinger quantization:

$$p^\mu \rightarrow i\hbar \partial^\mu \quad -(37)$$

i.e

$$\underline{p}\,\phi = -i\hbar \underline{\nabla}\phi \quad -(38)$$

to eq. (34) to find:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = \frac{1}{2}(\gamma^2-1)mc^2\phi \quad -(39)$$

$$:= E_{rel}\,\phi$$

where

$$E_{rel} = \frac{1}{2}(\gamma^2-1)mc^2 \quad -(40)$$

$$= \frac{1}{2}\left(\left(1-\left(\frac{p_0}{mc}\right)^2\right)^{-1} - 1\right)mc^2$$

$$\xrightarrow[v \ll c]{} \frac{p_0^2}{2m}\phi$$

where

$$\underline{p_0} = m\underline{v} \quad -(41)$$

The solution of eq. (39) is:

$$\phi = A\exp(i\kappa Z) + B\exp(-i\kappa Z) \quad -(42)$$

where

$$\boxed{\kappa^2 = \frac{2mE_1}{\hbar^2}} \quad -(43)$$

8) So:

$$\boxed{\kappa^2 = \left(\frac{mc}{\hbar}\right)^2 \left(\left(1 - \left(\frac{p_0}{mc}\right)^2 \right)^{-1} - 1 \right)} \quad -(44)$$

For a free particle, p_0 is a constant, so eq. (44) can be tested experimentally by measuring κ and p_0. It is equivalent to:

$$E^2 = \hbar^2 \omega^2 = \hbar^2 \kappa^2 + m^2 c^4 \quad -(45)$$

so

$$\omega^2 = \kappa^2 + \left(\frac{mc}{\hbar}\right)^2 \quad -(46)$$

which can also be tested experimentally.

In the limit:

$$p_0 << mc \quad -(47)$$

eq. (44) goes to:

$$p_0 = \hbar \kappa \quad -(48)$$

Q.E.D.

Eq. (44) is, self consistently:

9)

$$\kappa^2 = \frac{m^2c^2}{\hbar^2}(\gamma^2 - 1) = \frac{p^2}{\hbar^2} \quad -(49)$$

where

$$p = \gamma m v \quad -(50)$$

so

$$p^2 = m^2c^2(\gamma^2 - 1) \quad -(51)$$

i.e.

$$p^2 = (E^2 - m^2c^4)/c^2 \quad -(52)$$

or

$$E^2 = c^2p^2 + m^2c^4 \quad -(53)$$

QED.

Finally, solving eq. (44) for p_0 gives:

$$\boxed{\left(\frac{p_0}{mc}\right)^2 = 1 - \frac{1}{1 + \left(\frac{\hbar\kappa}{mc}\right)^2}} \quad -(54)$$

which is the relativistic version of:

$$p_0 = \hbar\kappa \quad -(55)$$

QED.

In the limit:

$$\hbar\kappa \ll mc \quad -(56)$$

$$\left(1+\left(\frac{\hbar\kappa}{mc}\right)^2\right)^{-1} \rightarrow 1 - \left(\frac{\hbar\kappa}{mc}\right)^2 \quad -(57)$$

So in eq. (54):

$$\left(\frac{p_0}{mc}\right)^2 \rightarrow \left(\frac{\hbar\kappa}{mc}\right)^2 \quad -(58)$$

i.e.

$$p_0 \rightarrow \pm\hbar\kappa. \quad -(59)$$

QED

The main result is that the Compton wavelength equation is changed to eq. (44) for relativistic particles.

These calculations also apply to the free particle rotational motion, the relativistic particle on a ring. The next step is to repeat them for the relativistic hydrogen atom.