323(5): Theory of Orbits with the Lorentz Transformation

Consider the Lorentz transform of the acceleration due to gravity:

$$\underline{g}' = \gamma(\underline{g} + \underline{v} \times \underline{\Omega}) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{g}\right) \quad (1)$$

$$\underline{\Omega}' = \gamma\left(\underline{\Omega} - \frac{1}{c^2}\underline{v} \times \underline{g}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{\Omega}\right) \quad (2)$$

where $\underline{\Omega}$ is the gravitomagnetic field. The inverse Lorentz transform is: $\quad (3)$

$$\underline{g} = \gamma(\underline{g}' - \underline{v} \times \underline{\Omega}') - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{g}'\right) \quad (4)$$

$$\underline{\Omega} = \gamma\left(\underline{\Omega}' + \frac{1}{c^2}\underline{v} \times \underline{g}'\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{\Omega}'\right)$$

Here g' is the acceleration in the Newtonian frame in which the axes of the coordinate system are not rotating or translating. The unprimed quantities are defined in the observer frame, which in general may both rotate and translate with respect to the primed frame.

For a central inverse square force:

$$\underline{g}' = \left(-\frac{mG}{r^2}\underline{e}_r\right)' = \left(\frac{d^2 r}{dt^2}\underline{e}_r\right)' \quad (5)$$

2)

i.e.:

$$\left(m \frac{d^2 r}{dt^2} \underline{e}_r \right)' = - \frac{mMG}{r^2} \underline{e}_r \quad -(6)$$

$$= m \left(\gamma \left(\underline{g} + \underline{v} \times \underline{\Omega} \right) - \frac{\gamma^2}{1+\gamma} \frac{\underline{v}}{c} \left(\frac{\underline{v}}{c} \cdot \underline{g} \right) \right)$$

This is the generalization of the Leibnitz equation of orbits to general relativity, ECE2 theory. Eq. (6) is valid in general for three dimensional orbits, and the two frames move with respect to each other with any velocity $\underline{v}$.

The mass current density of ECE2 theory is:

$$J^\mu = \left(c\rho, \underline{J} \right) \quad -(7)$$

and the gravitational four potential of ECE2 is:

$$a^\mu = \left(\frac{\Phi}{c}, \underline{a} \right) \quad -(8)$$

Under the general Lorentz transform:

$$\rho' = \gamma \left(\rho - \frac{1}{c^2} \underline{v} \cdot \underline{J} \right) \quad -(9)$$

$$\underline{J}' = \underline{J} - \gamma \rho \underline{v} + \frac{\gamma^2}{1+\gamma} \left(\underline{J} \cdot \frac{\underline{v}}{c} \right) \frac{\underline{v}}{c} \quad -(10)$$

Similarly:

5)

$$\Phi' = \gamma\left(\Phi - \underline{v}\cdot\underline{a}\right) \quad -(11)$$

and

$$\underline{a}' = \underline{a} - \frac{\gamma\Phi}{c^2}\underline{v} + \frac{\gamma^2}{1+\gamma}\left(\underline{a}\cdot\frac{\underline{v}}{c}\right)\frac{\underline{v}}{c} \quad -(12)$$

Using the minimal prescription:

$$p^\mu = m a^\mu \quad -(13)$$

eqns. (11) and (12) become the general Lorentz transform of the energy / momentum four vector:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(14)$$

which is:

$$E' = \gamma\left(E - \underline{v}\cdot\underline{p}\right) \quad -(15)$$

and

$$\underline{p}' = \underline{p} - \frac{\gamma E}{c^2}\underline{v} + \frac{\gamma^2}{1+\gamma}\left(\underline{p}\cdot\frac{\underline{v}}{c}\right)\frac{\underline{v}}{c} \quad -(16)$$

For a free particle:

$$E = \gamma m c^2, \quad \underline{p} = \gamma m \underline{v} \quad -(17)$$

so

$$\Phi = \gamma c^2, \quad \underline{a} = \gamma \underline{v} \quad -(18)$$

Therefore

$$p^\mu = \gamma m\left(c, \underline{v}\right) \quad -(19)$$

From eqns (11) and (18):

4)

$$\underline{\Phi}' = \gamma^2(c^2 - v^2) \quad -(20)$$

$$= c^2 .$$

So if

$$\underline{\Phi} = \gamma c^2 \quad -(21)$$

then:

$$\underline{\Phi}' = c^2 \quad -(22)$$

and

$$\gamma = \frac{\underline{\Phi}}{\underline{\Phi}'} = \frac{dt}{d\tau} \quad -(23)$$

where $d\tau$ is the infinitesimal of proper time.

Similarly, from eqs. (12) and (18):

$$\underline{Q}' = \left(\frac{\gamma^3}{1+\gamma}\right)\frac{v^2}{c^2}\,\underline{v} \quad -(24)$$

where:

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = \frac{\gamma^2 - 1}{\gamma^2} \quad -(25)$$

so

$$\underline{Q}' = \gamma(\gamma - 1)\,\underline{v} = (\gamma - 1)\underline{Q} \quad -(26)$$

for a free particle.

From eqs. (13), (23) and (26):

$$\gamma = \frac{E}{E'} \quad -(27)$$

and

$$\underline{p}' = (\gamma - 1)\,\underline{p} \quad -(28)$$

Therefore:

$$E = \gamma E' \quad -(29)$$

and

$$\underline{p}' = (\gamma - 1)\underline{p} \quad -(30)$$

In the non relativistic limit:

$$\gamma \rightarrow 1 \quad -(31)$$

so

$$\underline{p}' = \underline{0} \quad -(32)$$

This shows that the primed frame is the one in which the velocity is zero in the non-relativistic limit. In the primed frame E' is the rest energy:

$$E' = mc^2 \quad -(33)$$

So

$$E = \gamma mc^2 \quad -(34)$$

as in eq. (17), Q.E.D.

For a central potential in gravitation theory:

$$\Phi' = -\frac{MG}{r} \quad -(36)$$

so it is possible to use eqs (11) and (36) to establish relation between Φ and $\underline{Q}$:

$$-\frac{MG}{r} = \gamma\left(\Phi - \underline{v}\cdot\underline{Q}\right) \quad -(37)$$

In order to begin the development of eq. (6)

6) note that in the non-relativistic limit the fixed frame or primed frame equation is

$$\underline{F}' = m\underline{g}' = -\frac{mMG}{r^2}\underline{e}_r \quad -(38)$$

It is important to note that the prime means that the frame is not moving, and that the particle accelerates in this fixed frame. This is the Newtonian or inertial frame.

In a frame that moves with any velocity $\underline{v}$ with respect to the Newtonian frame the force is

$$\underline{F} = m(\underline{g} + \underline{v} \times \underline{\Omega}) = -\frac{mMG}{r^2}\underline{e}_r \quad -(39)$$

For plane polar coordinates:

$$\underline{F} = m\left((\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta\right) \quad -(40)$$

$$= -\frac{mMG}{r^2}\underline{e}_r$$

For planar orbits:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(41)$$

so

$$\underline{F} = m(\ddot{r} - r\dot{\theta}^2)\underline{e}_r \quad -(42)$$

Comparing eqs. (39) and (42):

$$\underline{g} + \underline{v} \times \underline{\Omega} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r \quad -(42)$$

so

$$\underline{g} = \ddot{r}\,\underline{e}_r \quad -(43)$$

and

$$\underline{v} \times \underline{\Omega} = \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(44)$$

i.e.

$$\underline{v} = \underline{\omega} \times \underline{r} \quad -(45)$$

and

$$\underline{\Omega} = -\underline{\omega} \quad -(46)$$

Therefore the velocity of the Lorentz transform for a planar orbit is orbital linear velocity:

$$\underline{v} = \underline{\omega} \times \underline{r} \quad -(47)$$

$$= \omega r\,\underline{e}_\theta$$

This is the angular part of the total velocity

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta. \quad -(48)$$

Therefore we arrive at the 1689 Leibnitz equation:

$$m\ddot{r}\,\underline{e}_r = -\frac{mMG}{r^2}\,\underline{e}_r + \omega \times (\underline{\omega} \times \underline{r}) \quad -(49)$$

and the elliptical or conic section orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(50)$$

For a planar orbit, eq. (1) becomes:

$$\underline{F} = \gamma\left(\ddot{r}\,\underline{e}_r - \omega^2 r\,\underline{e}_r\right) + \frac{\gamma^2}{1+\gamma}\,\frac{v}{c}\left(\frac{\underline{v}}{c}\cdot\underline{g}\right)$$

$$-(51)$$

where:

$$\underline{v} = \omega r \underline{e}_\theta, \quad \underline{g} = -\frac{MG}{r^2}\underline{e}_r \quad -(52)$$

So:

$$\underline{F} = m\gamma(\ddot{r} - \omega^2 r)\underline{e}_r = -\frac{mMG}{r^2}\underline{e}_r \quad -(53)$$

It follows that:

$$m(\ddot{r} - \omega^2 r) = -m\frac{MG}{\gamma r^2}. \quad -(54)$$

The velocity $\underline{v}$ of the Lorentz transform is $\underline{\omega} \times \underline{r}$ and this is part of:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad -(55)$$

The lagrangian for eq. (55) is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \quad -(56)$$

Eq. (56) gives:

$$m(\ddot{r} - \omega^2 r) = -\frac{mMG}{r^2} = -\frac{\partial U}{\partial r} \quad -(57)$$

and the Binet equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \quad -(58)$$

This suggests that the relativistic correction

9) is due to an effective potential:

$$F(r) = -\frac{\partial V(r)}{\partial r} = -\frac{mMG}{\gamma r^2} \quad -(59)$$

Therefore the orbit due to eq. (59) is given by:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{n^2 mGr^2}{\gamma r^2 L^2} = \frac{1}{\gamma\alpha} \quad -(60)$$

where α is the half right latitude and γ the Lorentz factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(61)$$

in which

$$v^2 = \omega^2 r^2 \quad -(62)$$

Integrating eq. (60) is not trivial, so it is easier to realize that the experimentally observed orbit is:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(63)$$

From eq. (58) this is generated by a force law:

$$F = mMG\left(-\frac{x^2}{r^2} + \frac{(x^2 - 1)\alpha}{r^3}\right) \quad -(64)$$

Comparing eqs. (59) and (64):

$$-\frac{x^2}{r^2} + \frac{(x^2-1)\alpha}{r^3} = -\frac{1}{\gamma r^2} \quad (65)$$

So

$$\boxed{x^2 + \frac{(x^2-1)\alpha}{r} = \frac{1}{\gamma}} \quad (66)$$

At the perihelion:

$$r = \frac{\alpha}{1+\epsilon} \quad (67)$$

So

$$x^2 + (x^2-1)(1+\epsilon) = \frac{1}{\gamma} \quad (68)$$

Therefore the Lorentz factor needed for the precession of the perihelion can be found within experimental precision.

Conclusion

It has been shown that the precession of the perihelion of a planar orbit is due to the general Lorentz transform of ECE2 theory.