323(1): Generation of Non-Newtonian Forces Using a Rotational Lorentz Transform.

The rotational Lorentz transform is defined by:

$$\begin{bmatrix} \underline{e}_0 \\ \underline{e}_r \\ \underline{e}_\theta \\ \underline{k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{e}_0 \\ \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(1)$$

From eq. (1):

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(2)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(3)$$

and

$$\underline{i} = \underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta \quad -(4)$$

$$\underline{j} = \underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta \quad -(5)$$

In a plane:

$$\underline{r} = X\underline{i} + Y\underline{j} = r\underline{e}_r \quad -(6)$$

and

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{r}\underline{e}_r + r\dot{\underline{e}}_r \quad -(7)$$

From eq. (2):

$$\dot{\underline{e}}_r = \underline{i}\frac{d\cos\theta}{dt} + \underline{j}\frac{d}{dt}\sin\theta \quad -(8)$$

$$= \underline{i}\left(\frac{d\cos\theta}{d\theta}\right)\frac{d\theta}{dt} + \underline{j}\left(\frac{d\sin\theta}{d\theta}\right)\frac{d\theta}{dt}$$

$$= \omega\left(-\sin\theta\, \underline{i} + \cos\theta\, \underline{j}\right)$$

$$= \omega\, \underline{e}_\theta$$

So:

$$\underline{\dot{e}}_r = \omega\, \underline{e}_\theta. \quad -(9)$$

Similarly:

$$\underline{\dot{e}}_\theta = -\omega\, \underline{e}_r. \quad -(10)$$

Therefore

$$\underline{v} = \dot{r}\, \underline{e}_r + \omega r\, \underline{e}_\theta \quad -(11)$$

$$= \dot{r}\, \underline{e}_r + \underline{\omega} \times \underline{r}.$$

Similarly:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta\right).$$

$$= \ddot{r}\,\underline{e}_r + \dot{r}\,\underline{\dot{e}}_r + \dot{r}\dot{\theta}\,\underline{e}_\theta + r\ddot{\theta}\,\underline{e}_\theta + r\dot{\theta}\,\underline{\dot{e}}_\theta \quad -(12)$$

in which

$$\dot{r}\,\underline{\dot{e}}_r = \dot{r}\dot{\theta}\,\underline{e}_\theta \quad -(13)$$

and

$$r\dot{\theta}\,\underline{\dot{e}}_\theta = -\omega^2 r\,\underline{e}_r \quad -(14)$$

where

$$\omega = \dot{\theta} \quad -(15)$$

so

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(16)$$

In vector notation eq. (16) can be expressed as follows:

3)

$$\underline{a} = \ddot{r}\underline{e}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r \quad -(17)$$

using:

$$\underline{e}_r = \underline{e}_\theta \times \underline{k} \quad -(18)$$

$$\underline{e}_\theta = \underline{k} \times \underline{e}_r \quad -(19)$$

$$\underline{k} = \underline{e}_r \times \underline{e}_\theta \quad -(20)$$

The centripetal acceleration is $-\underline{\omega} \times (\underline{\omega} \times \underline{r})$, the Coriolis acceleration is $2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r$ and there is a third acceleration $(d\underline{\omega}/dt) \times \underline{r}$. Eq. (17) follows from eq. (16) using:

$$\underline{\omega} \times \underline{r} = \omega \underline{k} \times r\underline{e}_r = \omega r \underline{e}_\theta \quad -(21)$$

$$\underline{\omega} \times \omega r \underline{e}_\theta = \omega^2 r \underline{k} \times \underline{e}_\theta = -\omega^2 r \underline{e}_r \quad -(22)$$

$$r\ddot{\theta}\underline{e}_\theta = \ddot{\theta}\underline{k} \times r\underline{e}_r = r\ddot{\theta}\underline{e}_\theta \quad -(23)$$

$$2\dot{r}\dot{\theta}\underline{e}_\theta = 2\omega\underline{k} \times \frac{dr}{dt}\underline{e}_r = 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r \quad -(24)$$

In a planar orbit:

$$\frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r = \underline{0} \quad -(25)$$

so

$$\underline{a} = \ddot{r}\underline{e}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(26)$$

The relevant acceleration is that due to gravity, so

4)
$$\underline{F} = m\underline{g} = m\left(\ddot{r}\,\underline{e}_r - \underline{\omega}\times(\underline{\omega}\times\underline{r})\right) = -\frac{mMG}{r^2}\,\underline{e}_r \qquad -(27)$$

from which we obtain the equation of orbits:

$$m\ddot{r}\,\underline{e}_r = -\frac{mMG}{r^2}\,\underline{e}_r + m\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) \qquad -(28)$$

The inverse square law of attraction is balanced by the centrifugal force $m\,\underline{\omega}\times(\underline{\omega}\times\underline{r})$.
It has been shown that the entire subject of non-Newtonian dynamics is generated by the rotational Lorentz transformation of unit vectors, eq. (1).
This suggests that the rotational Lorentz transform can also be applied in electrodynamics and gravitomagnetic theory. In electrodynamics, if

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \qquad -(29)$$

in S.I. units then the rotational Lorentz transform can be applied to produce $F^{\mu\nu\prime}$ as follows:

$$
= F^{\mu\nu'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(30)
$$

producing the results:

$$E_X' = E_X \cos\theta + E_Y \sin\theta \quad -(31)$$

$$E_Y' = -E_X \sin\theta + E_Y \cos\theta \quad -(32)$$

$$E_Z' = E_Z \quad -(33)$$

and

$$B_X' = B_Y \sin\theta + B_X \cos\theta \quad -(34)$$

$$B_Y' = B_Y \cos\theta - B_X \sin\theta \quad -(35)$$

$$B_Z' = B_Z \quad -(36)$$

Similarly, for the gravitomagnetic field tensor:

$$g_X' = g_X \cos\theta + g_Y \sin\theta \quad -(37)$$

$$g_Y' = -g_X \sin\theta + g_Y \cos\theta \quad -(38)$$

$$g_Z' = g_Z \quad -(39)$$

The gravitomagnetic field $\underline{\Omega}$ transforms according to:

6)

$$\Omega_x' = \Omega_y \sin\theta + \Omega_x \cos\theta \quad -(40)$$

$$\Omega_y' = \Omega_y \cos\theta - \Omega_x \sin\theta \quad -(41)$$

$$\Omega_z' = \Omega_z \quad -(42)$$

Now note that:

$$g_r = \ddot{r} - r\dot{\theta}^2 , \quad g_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad -(43)$$

and that:

$$\underline{g} = g_r \underline{e}_r + g_\theta \underline{e}_\theta$$
$$= g_x \underline{i} + g_y \underline{k} . \quad -(44)$$

Here:

$$\underline{i} = \underline{e}_r \cos\theta - \underline{e}_\theta \sin\theta \quad -(45)$$

$$\underline{j} = \underline{e}_r \sin\theta + \underline{e}_\theta \cos\theta \quad -(46)$$

so:

$$\underline{g} = (g_x \cos\theta + g_y \sin\theta)\underline{e}_r + (-g_x \sin\theta + g_y \cos\theta)\underline{e}_\theta \quad -(47)$$

and

$$g_r = g_x \cos\theta + g_y \sin\theta \quad -(48)$$

$$g_\theta = -g_x \sin\theta + g_y \cos\theta \quad -(49)$$

From eqs. (37), (38), (48) and (49):

$$\boxed{\begin{aligned} g_x' &= g_r \\ g_y' &= g_\theta \end{aligned}} \quad -(50)$$

7) It has been shown that the rotational Lorentz transform of the field tensor produces the same result as the rotational Lorentz transform of the unit vectors in eq. (1). The primes in eqs. (50) denote a frame of reference that is rotating. This rotating frame defines the cylindrical polar coordinate system, and if $Z = 0$ this is the plane polar coordinate system.

Similarly, the gravitomagnetic field is defined by:

$$\underline{\Omega} = \Omega_r \underline{e}_r + \Omega_\theta \underline{e}_\theta . \quad -(51)$$

where

$$\Omega_r = \Omega_X' = \Omega_X \cos\theta + \Omega_Y \sin\theta \quad -(52)$$

$$\Omega_\theta = \Omega_Y' = \Omega_Y \cos\theta - \Omega_X \sin\theta \quad -(53)$$

We now compare the results of the rotational Lorentz transform (30) with those of the Lorentz boost transform of the same field tensor. The results of the Lorentz boost transform are:

$$-(54)$$

$$\underline{g}' = \gamma\left(\underline{g} + \underline{v} \times \underline{\Omega}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{g}\right)$$

$$\underline{\Omega}' = \gamma\left(\underline{\Omega} - \frac{1}{c^2}\underline{v} \times \underline{g}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{\Omega}\right)$$

$$-(55)$$

where:

5)

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(56)$$

is the Lorentz factor. The latter is defined by the metric:

$$c^2 d\tau^2 = c^2 dt^2 - v^2 dt^2 \quad -(57)$$

So:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(58)$$

The velocity is defined by:

$$v^2 = \dot{r}^2 + \omega^2 r^2 \quad -(59)$$

from eq. (11), which is generated by a rotational Lorentz transform.

Therefore there is a link between the results of the rotational Lorentz transform and the Lorentz boost transform of the gravitomagnetic field tensor. The velocity $\underline{v}$ of the boost transform must be defined by eq. (11) in order to ensure that the boost transform gives the same results as the rotational transform. <u>It follows that the whole of dynamics and gravito-magnetic dynamics can be defined by these transforms.</u>

The primes on the left hand sides of eqs.

9) (54) and (55) denote the observer frame in which there is a finite velocity $\underline{v}$. The primes in eqs. (50), (52) and (53) denote the same observer frame, which is a rotating frame. It is the frame of the cylindrical polar coordinate system, which becomes the plane polar coordinate system when $z = 0$.

Therefore the overall result is:

$$\underline{g}' = g_r \underline{e}_r + g_\theta \underline{e}_\theta = \gamma (\underline{g} + \underline{v} \times \underline{\Omega}) - \frac{\gamma^2}{1+\gamma} \frac{\underline{v}}{c} \left(\frac{\underline{v}}{c} \cdot \underline{g} \right) \quad - (56)$$

and

$$\underline{\Omega}' = \Omega_r \underline{e}_r + \Omega_\theta \underline{e}_\theta = \gamma \left(\underline{\Omega} - \frac{1}{c^2} \underline{v} \times \underline{g} \right) - \frac{\gamma^2}{1+\gamma} \frac{\underline{v}}{c} \left(\frac{\underline{v}}{c} \cdot \underline{\Omega} \right) \quad - (57)$$

where

$$\underline{v} = v_r \underline{e}_r + v_\theta \underline{e}_\theta = \frac{dr}{dt} \underline{e}_r + \omega r \underline{e}_\theta \quad - (58)$$

is the same for both the rotational and boost transformations.

These results generalize those of previous notes and papers, which dealt with the special case of planar orbits and the Biot Savart and Ampère laws of gravitomagnetic dynamics. These special cases are summarized here for ease of reference.

1) Planar Orbits

In this case:

$$\underline{F} = m\underline{g}' = m(\underline{g} - \underline{\omega} \times (\underline{\omega} \times \underline{r})) = -\frac{mMG}{r^2} \quad -(59)$$

so

$$\underline{g}' = \underline{g} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(60)$$

$$= \underline{g} + \underline{v} \times \underline{\omega}$$

where

$$\underline{v} = \underline{\omega} \times \underline{r} \quad -(61)$$

These are the results of the rotational Lorentz transform. The velocity $\underline{v}$ of eq. (58) is simplified to:

$$\underline{v} = v_\theta \, \underline{e}_\theta = \underline{\omega} \times \underline{r} \quad -(62)$$

The Lorentz boost is considered in the non-relativistic limit, so:

$$\gamma \rightarrow 1 \quad -(63)$$

and

$$v << c \quad -(64)$$

so:

$$\underline{g}' = \underline{g} + \underline{v} \times \underline{\Omega} \quad -(65)$$

It follows that

$$\underline{\Omega} = \underline{\omega} = \frac{L}{mr^2}\,\underline{k} \quad -(66)$$

where L is the conserved angular momentum.

2) Biot Savart Gravitomagnetic Law

This is a special case of eq. (57) where the metric is non-relativistic and where $\underline{\Omega}$ in the rest frame of a mass is zero. The velocity of the mass in its own rest frame is zero. The velocity in the observer frame is non-zero.

So:

$$\begin{aligned}\underline{\Omega}' &= \Omega_r\,\underline{e}_r + \Omega_\theta\,\underline{e}_\theta \\ &= -\frac{1}{c^2}\,\underline{v} \times \underline{g}\end{aligned} \quad -(68)$$

where

$$\underline{v} = v_r\,\underline{e}_r + v_\theta\,\underline{e}_\theta \quad -(69)$$

in general. For general dynamics $\underline{g}$ is replaced by the general acceleration of eq. (16). For the central inverse square force of an orbit:

$$\underline{F} = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(70)$$

the acceleration due to gravity is:

$$\underline{g} = -\frac{MG}{r^2}\underline{e}_r \quad -(71)$$

and

$$\underline{\Omega}' = \frac{1}{c^2} V_\theta\, g\, \underline{k} \quad -(72)$$

where

$$V_\theta = \omega r, \quad g = -\frac{MG}{r^2} \quad -(73)$$

The Ampère law is defined by:

$$\underline{\nabla} \times \underline{\Omega}' = -\frac{1}{c^2}\underline{\nabla} \times (\underline{v} \times \underline{a})$$

$$= \frac{4\pi G}{c^2}\underline{J}_m \quad -(74)$$

so the current of mass density is defined by:

$$\boxed{\underline{J}_m := -\frac{1}{4\pi G}\underline{\nabla} \times (\underline{v} \times \underline{a})} \quad -(75)$$

where

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(76)$$

and:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(77)$$