323(3): The Lorentz Boost in Z

Consider the position vector:

$$x^\mu = (ct, X, Y, Z) \quad -(1)$$

The Lorentz boost in Z is:

$$\begin{bmatrix} ct' \\ X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ X \\ Y \\ Z \end{bmatrix} \quad -(2)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

$$\beta = \frac{v}{c}. \quad -(4)$$

Therefore:

$$t' = \gamma\left(t - \frac{vZ}{c^2}\right) \quad -(5)$$

$$X' = X \quad -(6)$$

$$Y' = Y \quad -(7)$$

$$Z' = \gamma(Z - vt) \quad -(8)$$

Note that

$$x^\mu x_\mu = x^{\mu'} x_{\mu'} \quad -(9)$$

$$= c^2t^2 - X^2 - Y^2 - Z^2$$

is an invariant of the transformation. This result follows from the fact that:

d)

$$\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(10)$$

Therefore: $\quad -(11)$

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2$$

and in terms of infinitesimals:

$$c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad -(12)$$

in which:

$$v'^2 dt'^2 = dx'^2 + dy'^2 + dz'^2 \quad -(13)$$

and

$$v^2 dt^2 = dx^2 + dy^2 + dz^2 \quad -(14)$$

so

$$c^2dt'^2 - v'^2dt'^2 = c^2dt^2 - v^2dt^2 \quad -(15)$$

However, in the primed frame, the particle is defined to be moving in a frame fixed to the particle, so:

$$v'^2 = 0 \quad -(16)$$

The infinitesimal of proper time is defined as

$$d\tau^2 = dt'^2 \quad -(17)$$

So:

3)
$$c^2 d\tau^2 = (c^2 - v^2) dt^2 \quad - (18)$$

which is the Minkowski metric. In plane polar coordinates:
$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \quad - (19)$$

From eq. (18):
$$d\tau^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2 \quad - (20)$$

and the Lorentz factor is:
$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (21)$$

Now consider the Z axis boosts of the following four vectors, respectively the energy momentum, the potential four vector, and the charge current density:
$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad - (22)$$
$$A^\mu = \left(\frac{\phi}{c}, \underline{A}\right) \quad - (23)$$
$$J^\mu = \left(c\rho, \underline{J}\right) \quad - (24)$$

The four derivative may also be boosted in Z:
$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad - (25)$$

For example:

4) $$\begin{bmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \quad -(26)$$

So:

$$E' = \gamma(E - v p_z) \quad -(27)$$

$$p_x' = p_x \quad -(28)$$

$$p_y' = p_y \quad -(29)$$

$$p_z' = \gamma\left(p_z - \frac{v_z E}{c^2}\right) \quad -(30)$$

Similarly:

$$\phi' = \gamma(\phi - v_z A_z) \quad -(31)$$

$$A_z' = \gamma\left(A_z - \phi\frac{v_z}{c^2}\right) \quad -(32)$$

$$\rho' = \gamma\left(\rho - \frac{v_z J_z}{c^2}\right) \quad -(33)$$

$$J_z' = \gamma(J_z - \rho v_z) \quad -(34)$$

According to the Wikipedia website these results are generalized for any boost with general velocity:

$$\underline{v} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k} \quad -(35)$$

5) to:

$$\rho' = \gamma\left(\rho - \frac{1}{c^2}\underline{v}\cdot\underline{J}\right) - (36)$$

$$\underline{J}' = \underline{J} - \gamma\rho\underline{v} + (\gamma-1)(\underline{J}\cdot\hat{\underline{v}})\hat{\underline{v}} - (37)$$

and

$$\phi' = \gamma\left(\phi - \underline{v}\cdot\underline{A}\right) - (38)$$

$$\underline{A}' = \gamma\left(\underline{A} - \frac{\phi}{c^2}\underline{v}\right) + (\gamma-1)(\underline{A}\cdot\hat{\underline{v}})\hat{\underline{v}} - (39)$$

The General Lorentz Boost

This is evaluated by using:

$$\underline{r} = \underline{r}_{\parallel} + \underline{r}_{\perp} - (40)$$

where

$$\underline{r}_{\parallel} \parallel \underline{v} - (41)$$

So

$$\underline{r}\cdot\underline{v} = \underline{r}_{\perp}\cdot\underline{v} + \underline{r}_{\parallel}\cdot\underline{v} = r_{\parallel} v - (42)$$

Therefore:

$$t' = \gamma\left(t - \frac{1}{c^2}\underline{r}\cdot\underline{v}\right) - (43)$$

$$\underline{r}' = \underline{r}_{\perp} + \gamma\left(\underline{r}_{\parallel} - t\underline{v}\right) - (44)$$

Now use:

$$\underline{r}_{\perp} = \underline{r} - \underline{r}_{\parallel} - (45)$$

6)

So:
$$\underline{r}' = \underline{r} + (\gamma - 1)\underline{r}_{\parallel} - \gamma \underline{v} t \quad -(46)$$

Since
$$\underline{r}_{\parallel} \parallel \underline{v} \quad -(47)$$

then:
$$\underline{r}_{\parallel} = r_{\parallel} \frac{\underline{v}}{v} = \left(\frac{\underline{r} \cdot \underline{v}}{v} \right) \frac{\underline{v}}{v} \quad -(48)$$

so
$$\underline{r}' = \underline{r} + \left(\frac{\gamma - 1}{v^2} \underline{r} \cdot \underline{v} - \gamma t \right) \underline{v} \quad -(49)$$

The most general Lorentz boost matrix is:

$$\Lambda^{\nu'}_{\mu} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + (\gamma - 1)\left(\frac{\beta_x}{\beta}\right)^2 & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix} \quad -(50)$$

where:
$$\beta_x = \frac{v_x}{c}, \quad \beta_y = \frac{v_y}{c}, \quad \beta_z = \frac{v_z}{c} \quad -(51)$$

and
$$\beta = \frac{v}{c} \quad -(52)$$

It should be checked by computer algebra that eq. (50) gives eqs. (36) to (39), i.e:

7)

$$J^{\nu'} = \Lambda^{\nu}_{\mu} J^{\mu} \quad -(53)$$

and

$$A^{\nu'} = \Lambda^{\nu}_{\mu} J^{\mu} \quad -(54)$$

<u>Non Relativistic Limit</u>

In this case: $v << c, \gamma \to 1 \quad -(55)$

So:

$$\rho' = \rho - \frac{1}{c^2} \underline{v} \cdot \underline{J} \quad -(56)$$

$$\underline{J}' = \underline{J} - \rho \underline{v} \quad -(57)$$

$$\phi' = \phi - \underline{v} \cdot \underline{A} \quad -(58)$$

$$\underline{A}' = \underline{A} - \frac{\phi}{c^2} \underline{v} \quad -(59)$$

The inverse results are:

$$\rho = \rho' + \frac{1}{c^2} \underline{v} \cdot \underline{J}' \quad -(60)$$

$$\underline{J} = \underline{J}' + \rho' \underline{v} \quad -(61)$$

$$\phi = \phi' + \underline{v} \cdot \underline{A}' \quad -(62)$$

$$\underline{A} = \underline{A}' + \frac{\phi'}{c^2} \underline{v} \quad -(63)$$

The inverse transforms are obtained from the

8) Lorentz transform by reversing the sign of $\underline{v}$ and by interchanging the primed and unprimed labels.

The primed frame is the frame in which the velocity is zero if a particle is assumed to move with the primed frame. The unprimed frame is the frame in which the velocity $\underline{v}$ of the particle is non-zero. From eq. (57) for example, if:

$$\underline{J} = \rho \underline{v} \quad -(64)$$

then:

$$\underline{J}' = \underline{0} \quad -(65)$$

Eq. (64) is the usual solution of

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0 \quad -(66)$$

which in electrodynamics is the equation of conservation of charge, and in dynamics is the equation of conservation of matter. The current $\underline{J}'$ is zero in the primed frame because $\underline{v}$ is zero in the primed frame. From eq. (60) and (65):

$$\rho = \rho' \quad -(67)$$

i.e. the charge or mass density is invariant under the Lorentz transform. From eq. (61):

$$\underline{J} = \rho \underline{v} = \rho' \underline{v}' \quad -(62)$$

<u>The Z Boost</u> and <u>Transformation of the Field Tensor</u>

The field tensor is:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix} \quad -(63)$$

The Z boost transform gives the field tensor:

$$F^{\mu'\nu'} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \quad -(64)$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \gamma v E_z/c & -E_x & -E_y & -\gamma E_z \\ \gamma(E_x - vB_y) & 0 & -cB_z & \gamma(cB_y - vE_x/c) \\ \gamma(E_y + vB_x) & cB_z & 0 & -\gamma(cB_x + vE_y/c \\ \gamma E_z & -cB_y & cB_x & -\gamma v E_z/c \end{bmatrix} \quad -(65)$$

$$= \begin{bmatrix} 0 & -\gamma(E_x - vB_y) & -\gamma(E_y + vB_x) & -E_z \\ \gamma(E_x - vB_y) & 0 & -cB_z & \gamma(cB_y - vE_x/c) \\ \gamma(E_y + vB_x) & cB_z & 0 & -\gamma(cB_x + vE_y/c \\ E_z & -\gamma(cB_y - vE_x/c) & \gamma(cB_x + vE_y/c) & 0 \end{bmatrix}$$

10) So:

$$E_z' = E_z \,, \quad B_z' = B_z \quad -(66)$$

$$E_x' = \gamma(E_x - v_z B_y) \,, \quad B_y' = \gamma\left(B_y - \frac{v_z}{c^2} E_x\right) \quad -(67)$$

$$E_y' = \gamma(E_y + v_z B_x) \quad B_x = \gamma\left(B_x + \frac{v_z E_y}{c^2}\right) \quad -(68)$$

It is seen that eqs. (67) and (68) are components of

$$\underline{E}' = \gamma(\underline{E} + \underline{v} \times \underline{B}) \quad -(69)$$

and

$$\underline{B}' = \gamma\left(\underline{B} - \frac{1}{c^2}\underline{v} \times \underline{E}\right) \quad -(70)$$

Using the general matrix (50), Jackson (3rd. ed.) gives the result:

$$\underline{E}' = \gamma(\underline{E} + \underline{v} \times \underline{B}) - \frac{\gamma^2}{\gamma + 1}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{E}\right) \quad -(71)$$

and $$\underline{B}' = \gamma\left(\underline{B} - \frac{1}{c^2}\underline{v} \times \underline{E}\right) - \frac{\gamma^2}{\gamma + 1}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{B}\right) \quad -(72)$$

However the wiki site "Classical E/M and Special Relativity" gives:

$$\underline{E}' = \gamma(\underline{E} + \underline{v} \times \underline{B}) - (\gamma - 1)\frac{\underline{v}}{v}\left(\frac{\underline{v}}{v} \cdot \underline{E}\right) \quad -(73)$$

$$\underline{B}' = \gamma\left(\underline{B} - \frac{1}{c^2}\underline{v} \times \underline{E}\right) - (\gamma - 1)\frac{\underline{v}}{v}\left(\frac{\underline{v}}{v} \cdot \underline{B}\right) \quad -(74)$$

). Using:

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(75)$$

it is found that eqs. (71) and (72) are the same as eqs. (73) and (74). It should be checked by computer algebra that the general matrix (50) gives the result (71).

The inverse of eqs. (71) and (72) is:

$$\underline{E} = \gamma\left(\underline{E}' - \underline{v} \times \underline{B}'\right) - \frac{\gamma^2}{\gamma+1}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{E}'\right) \quad -(76)$$

and

$$\underline{B} = \gamma\left(\underline{B}' + \frac{1}{c^2}\underline{v} \times \underline{E}'\right) - \frac{\gamma^2}{\gamma+1}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{B}'\right) \quad -(77)$$

In the non relativistic limit eqs. (69) and (70) are found, with

$$\gamma \to 1 \quad -(78)$$

So:

$$\underline{E}' = \underline{E} + \underline{v} \times \underline{B} \quad -(79)$$

$$\underline{B}' = \underline{B} - \frac{1}{c^2}\underline{v} \times \underline{E} \quad -(80)$$

$$\underline{E} = \underline{E}' - \underline{v} \times \underline{B}' \quad -(81)$$

$$\underline{B} = \underline{B}' + \frac{1}{c^2}\underline{v} \times \underline{E}' \quad -(82)$$