323(2): ECE2 Dynamics

In note 323(1) it was shown that for planar orbits, the Lorentz rotation of the unit four vector is equivalent to the Lorentz boost of the field tensor. In this case the Lorentz boost velocity is $\underline{v} = \underline{\omega} \times \underline{r}$ and $\underline{\omega} = \omega \underline{k}$, $\underline{r} = r \underline{e}_r$.

In this note it is proposed that dynamics in general are governed by the ECE2 equations:

$$\partial_\mu \tilde{G}^{\mu\nu} = 0 \quad -(1)$$

and

$$\partial_\mu G^{\mu\nu} = J^\nu \quad -(2)$$

where:

$$G^{\mu\nu} = \begin{bmatrix} 0 & -a^1 & -a^2 & -a^3 \\ a^1 & 0 & -c\Omega^3 & c\Omega^2 \\ a^2 & c\Omega^3 & 0 & -c\Omega^1 \\ a^3 & -c\Omega^2 & c\Omega^1 & 0 \end{bmatrix} \quad -(3)$$

and

$$\tilde{G}^{\mu\nu} = \begin{bmatrix} 0 & -c\Omega^1 & -c\Omega^2 & -c\Omega^3 \\ c\Omega^1 & 0 & a^3 & -a^2 \\ c\Omega^2 & -a^3 & 0 & a^1 \\ c\Omega^3 & a^2 & -a^1 & 0 \end{bmatrix} \quad -(4)$$

Most generally, the equations of dynamics are:

2)

$$\underline{\nabla} \cdot \underline{a} = \underline{\kappa} \cdot \underline{a} = 4\pi G \rho_m \quad - (5)$$

$$\underline{\nabla} \times \underline{\Omega} - \frac{1}{c^2} \frac{\partial \underline{a}}{\partial t} = \frac{\kappa_0}{c} \underline{a} + \underline{\kappa} \times \underline{\Omega} = \frac{4\pi G}{c^2} \underline{J}_m \quad - (6)$$

$$\underline{\nabla} \cdot \underline{\Omega} = \underline{\kappa} \cdot \underline{\Omega} = \frac{4\pi G}{c} \rho_\Omega \quad - (7)$$

$$\underline{\nabla} \times \underline{a} + \frac{\partial \underline{\Omega}}{\partial t} = -(c\kappa_0 \underline{\Omega} + \underline{\kappa} \times \underline{a}) = \frac{4\pi G}{c^2} \underline{J}_\Omega \quad - (8)$$

$$\kappa_0 = 2\left(\frac{q_0}{r^{(0)}} - \omega_0\right) \quad - (9)$$

$$\underline{\kappa} = 2\left(\frac{\underline{q}}{r^{(0)}} - \underline{\omega}\right) \quad - (10)$$

In deriving eqs. (1) and (2) it has been assumed that eqs. (7) and (8) simplify to:

$$\underline{\nabla} \cdot \underline{\Omega} = 0 \quad - (11)$$

$$\underline{\nabla} \times \underline{g} + \frac{\partial \underline{\Omega}}{\partial t} = \underline{0} \quad - (12)$$

This is equivalent to the assumption in electrodynamics that there are no magnetic charge and currents.

3) From eqs. (5) and (6):

$$\frac{4\pi G}{c^2}\underline{\nabla}\cdot\underline{J}_m = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(\underline{\nabla}\cdot\underline{a}\right) = -\frac{4\pi G}{c^2}\frac{\partial \rho_m}{\partial t} \quad -(13)$$

i.e.

$$\frac{\partial \rho_m}{\partial t} + \underline{\nabla}\cdot\underline{J}_m = 0 \quad -(14)$$

This is the equation of conservation of matter, the continuity equation. Therefore eq. (2) is the most general form of the continuity equation.

In this notation $\underline{a}$ is the acceleration, $\underline{\Omega}$ is a quantity new to dynamics, ρ_m is the mass density and $\underline{J}_m$ is the current density, the density of mass current. In gravitational theory $\underline{\Omega}$ is the gravitomagnetic field.

The Lorentz boost with linear velocity $\underline{v}$ of the field tensor (3) gives:

$$\underline{a}' = \gamma\left(\underline{a} + \underline{v}\times\underline{\Omega}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{a}\right) \quad -(15)$$

$$\underline{\Omega}' = \gamma\left(\underline{\Omega} - \frac{1}{c^2}\underline{v}\times\underline{a}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{\Omega}\right) \quad -(16)$$

where γ is the Lorentz factor.

$$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \quad -(17)$$

Note carefully that eqs. (15) and (16) are those of a generally covariant unified field theory. They are equations of general relativity.

The primed frame moves with respect to the observer frame, and the primed frame is the rest frame of a particle moving in the observer frame. In the non-relativistic limit:

$$\gamma \rightarrow 1, \quad v \ll c \quad -(18)$$

so

$$\underline{a}' = \underline{a} + \underline{v} \times \underline{\Omega} \quad -(19)$$

$$\underline{\Omega}' = \underline{\Omega} - \frac{1}{c^2} \underline{v} \times \underline{a} \quad -(20)$$

The acceleration in the observer frame is therefore:

$$\underline{a} = \underline{a}' - \underline{v} \times \underline{\Omega} \quad -(21)$$

and similarly:

$$\underline{\Omega} = \underline{\Omega}' + \frac{1}{c^2} \underline{v} \times \underline{a} \quad -(22)$$

If the rest frame is interpreted as the Newtonian or inertial frame then:

$$\underline{a}' = \underline{a}_N, \quad \underline{\Omega}' = \underline{\Omega}_N \quad -(23)$$

$$\underline{a} = \underline{a}_N + \underline{\Omega} \times \underline{v} \quad -(24)$$

and

$$\underline{\Omega} = \underline{\Omega}_N + \frac{1}{c^2} \underline{v} \times \underline{a} \quad -(25)$$

If $\underline{\Omega}$ is interpreted as the angular velocity $\underline{\omega}$ of one frame with respect to another, then eq. (24) is the fundamental kinematic equation:

$$\underline{a} = \underline{a}_N + \underline{\omega} \times \underline{v} \quad -(26)$$

in which the Newtonian acceleration is:

$$\underline{a}_N = \frac{d^2 r}{dt^2} \underline{e}_r \quad -(27)$$

If one frame is rotated with respect to another at angular velocity

$$\underline{\omega} = \frac{d\theta}{dt} \underline{k} \quad -(28)$$

then as in note 323(1):

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(29)$$

and

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(30)$$

From eqs. (24) and (29) $\underline{\Omega}$ can be evaluated for general dynamics from:

6)

$$\underline{a}_N = \ddot{r}\,\underline{e}_r \quad -(31)$$

and:

$$\underline{\Omega} \times \underline{v} = \underline{\Omega} \times (\dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta) \quad -(32)$$

$$= -r\dot{\theta}^2\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta$$

In three dimensions:

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta + \dot{z}\,\underline{k} \quad -(33)$$

and

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta + \ddot{z}\,\underline{k} \quad -(34)$$

One possible solution of eq. (32) is:

$$\underline{\Omega} = \omega\,\underline{k} = \dot{\theta}\,\underline{k} \quad -(35)$$

and

$$r\ddot{\theta} + \dot{r}\dot{\theta} = 0 \quad -(36)$$

Conclusion

In general it is possible to describe dynamics by:

$$\underline{a}' = \underline{a} + \underline{v} \times \underline{\Omega} \quad -(37)$$

i.e.

$$\boxed{\underline{a} = \underline{a}_N + \underline{\Omega} \times \underline{v}} \quad -(38)$$

This is a generalization of the well known kinematic result:

$$\underline{a} = \underline{a}_N + \underline{\omega} \times \underline{v} \quad -(39)$$

Eq. (15) can also be interpreted as: $\quad -(40)$

$$\underline{a} = \gamma\left(\underline{a}' - \underline{v} \times \underline{\Omega}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{a}'\right)$$

by reversing the sign of $\underline{v}$.

So the observer frame acceleration is: $\quad -(41)$

$$\underline{a} = \gamma\left(\underline{a}_N - \underline{v} \times \underline{\Omega}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{a}_N\right)$$

where

$$\underline{a}_N = \frac{d^2 r}{dt^2}\,\underline{e}_r \quad -(42)$$

Eq. (41) is the relativistic generalization of the fundamental kinematic equation (39).