322(4): The Gravitomagnetic Field in General Dynamics

Consider the position coordinate:

$$\underline{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} \quad -(1)$$

In cylindrical polar coordinates:

$$X = r\cos\theta, \quad Y = r\sin\theta, \quad -(2)$$

So:

$$\underline{r} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + Z\,\underline{k} \quad -(3)$$

$$= r\,\underline{e}_r + Z\,\underline{k}.$$

With reference to "Vector Analysis Problem Solver", problem 21-12, p. 1029, the velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta + \dot{Z}\,\underline{k} \quad -(4)$$

and the acceleration is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta + \ddot{Z}\,\underline{k} \quad -(5)$$

where by definition:

$$\underline{\omega} = \frac{d\theta}{dt}\,\underline{k} \quad -(6)$$

Therefore the general gravitomagnetic field is:

$$\underline{\Omega} = -\frac{1}{c^2}\,\underline{v} \times \underline{a} \quad -(7)$$

$$= -\frac{1}{c^2}\begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ \dot{r} & \omega r & \dot{Z} \\ (\ddot{r} - r\dot{\theta}^2) & (r\ddot{\theta} + 2\dot{r}\dot{\theta}) & \ddot{Z} \end{vmatrix}$$

2) In general, this expression contains the three
non Newtonian forces considered in eq. (5).

Three Dimensional Orbits

These are defined by the central force:

$$\underline{F} = -\frac{mMG}{r^2}\underline{e}_r \quad -(8)$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}mv^2 - U \quad -(9)$$

where:

$$v^2 = \dot{r}^2 + \dot{\theta}^2 r^2 + \dot{Z}^2 \quad -(10)$$

and

$$U = -\frac{mG}{r} \quad -(11)$$

The three Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(12)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(13)$$

$$\frac{\partial \mathcal{L}}{\partial Z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Z}} \quad -(14)$$

Eq. (12) gives:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad -(15)$$

3) Eq. (13) gives:

$$F(r) = -\frac{\partial U}{\partial r} = m(\ddot{r} - r\dot{\theta}^2) \quad -(16)$$

Eq. (14) gives:

$$\frac{dZ}{dt} = 0. \quad -(17)$$

Eq. (15) is conservation of angular momentum, eq. (16) is the 1689 Leibniz equation of orbits, and eq. (17) shows that Z does not change with time.

The angular momentum is defined by:

$$\underline{L} = m\,\underline{r} \times \underline{v} \quad -(18)$$

$$= m \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ r & 0 & Z \\ \dot{r} & \omega r & 0 \end{vmatrix}$$

therefore:

$$\boxed{\underline{L} = m(\dot{r} Z\, \underline{e}_\theta + \omega r^2\, \underline{k})} \quad -(19)$$

an equation which shows clearly that $\underline{L}$ is not perpendicular to the orbital plane.

From eq. (19):

4)

$$L_Z = mr^2\omega \quad -(20)$$

$$L_\theta = m\dot{r}\cdot Z \quad -(21)$$

It follows that:

$$L_Z = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\omega \quad -(22)$$

and

$$\boxed{\omega_Z = \frac{L_Z}{mr^2} = \frac{d\theta}{dt}} \quad -(23)$$

The Lagrangian analysis shows that:

$$\frac{dL_Z}{dt} = 0 \quad -(24)$$

Therefore L_Z is a constant of motion.

Note carefully that:

$$\frac{dL}{dt} \neq 0 \quad -(25)$$

where

$$L^2 = L_\theta^2 + L_Z^2 \quad -(26)$$

From eqs. (16) and (23) it follows as in Marion and Thornton pages 249 ff of the third edition that:

5)
$$F(r) = -\frac{L_z^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(27)$$

which is the Binet equation of orbits

By definition:
$$\omega_z = \dot{\theta} = \frac{L_z}{mr^2}, \quad -(28)$$

so
$$\ddot{\theta} = \frac{d\dot{\theta}}{dr}\frac{dr}{dt} = \dot{r}\frac{d}{dr}\left(\frac{L_z}{mr^2}\right)$$
$$= -\frac{2\dot{r}L_z}{mr^3} \quad -(29)$$

Therefore:
$$r\ddot{\theta} = -\frac{2\dot{r}L_z}{mr^2}. \quad -(30)$$

Similarly:
$$2\dot{r}\dot{\theta} = \frac{2\dot{r}L_z}{mr^2}. \quad -(31)$$

It follows that:
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(32)$$

From eqs. (4), (5), (17) and (32):

$$\boxed{\begin{aligned} \underline{v} &= \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(33) \\ \underline{a} &= (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r \quad -(34) \end{aligned}}$$

6) and:

$$\underline{r} = r\,\underline{e}_r + Z\,\underline{k} \quad -(35)$$

$$\underline{L} = m\left(\dot{r}Z\,\underline{e}_\theta + \omega r^2\,\underline{k}\right) \quad -(36)$$

The difference between planar and three dimensional orbital theory is that in planar theory:

$$\underline{r} = r\,\underline{e}_r \quad -(37)$$

$$\underline{L} = m\omega r^2\,\underline{k} \quad -(38)$$

and

In planar theory:

$$Z = 0. \quad -(39)$$

For clarity eq. (35) can be defined as:

$$\underline{r}_{total} = r\,\underline{e}_r + Z\,\underline{k} \quad -(40)$$

so

$$r_{total}^2 = r^2 + Z^2. \quad -(41)$$

In general, the observed orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(42)$$

where

$$r = \left(r_{total}^2 - Z^2\right)^{1/2} \quad -(43)$$

and where

$$L_Z^2 = m^2 M G \alpha \quad -(44)$$

The angular velocity, gravitomagnetic field, and the mass current remain as in Note 322(2):

$$\underline{\omega} = \frac{L_Z}{mr^2}\,\underline{k} \quad -(45)$$

$$\underline{\lambda} = -\left(\frac{MG}{mc^2}\right)\frac{1}{r^3}\,\underline{k} \quad -(46)$$

$$\underline{J}_m = -\frac{3M}{4\pi m}\,\frac{L_Z}{r^4}\,\underline{e}_\theta \quad -(47)$$

with L_Z taking the place of L.

Finally, note that:

$$\underline{\omega} \times \underline{r} = \omega_Z\,\underline{k} \times (r\,\underline{e}_r + Z\,\underline{k})$$

$$= \omega_Z r\,\underline{k} \times \underline{e}_r \quad -(48)$$

$$= \omega_Z r\,\underline{e}_\theta$$

Therefore:

$$\underline{F} = m\underline{g} = -\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(49)$$

The added information is a 3-D orbit is:

$$\boxed{r_{total}^2 = \left(\frac{\alpha}{1+\epsilon\cos(x\theta)}\right)^2 + Z^2} \quad -(50)$$

9) In the most general dynamics:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 + \dot{Z}^2 \right) - U(r, \theta, Z) \quad -(51)$$

and in the most general dynamics:

$$\underline{a} = \ddot{r} \, \underline{e}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2 \underline{\omega} \times \frac{dr}{dt} \underline{e}_r \quad -(52)$$

in which the centripetal acceleration is $-\underline{\omega} \times (\underline{\omega} \times \underline{r})$, the Coriolis acceleration is $2 \underline{\omega} \times \frac{dr}{dt} \underline{e}_r$ and the third non-Newtonian acceleration is $\frac{d\underline{\omega}}{dt} \times \underline{r}$. The velocity in general dynamics is:

$$\underline{v} = \dot{r} \, \underline{e}_r + \underline{\omega} \times \underline{r} \quad -(53)$$

and the most general gravitomagnetic field is

$$\boxed{\underline{\Omega} = -\frac{1}{c^2} \underline{v} \times \underline{a}} \quad -(54)$$

Therefore $\underline{\Omega}$ must be worked out from eqs (52) and (53).