315(2): Some Notes on the Development of the First Evans Identity.

The first Evans identity is an exact identity of tensor algebra in all spaces of all dimensions:

$$T^{\lambda}_{\mu\nu} T^{\alpha}_{\rho\lambda} + T^{\lambda}_{\rho\mu} T^{\alpha}_{\nu\lambda} + T^{\lambda}_{\nu\rho} T^{\alpha}_{\mu\lambda} := 0 \quad -(1)$$

To Prove

Eq. (1) can be written as:

$$T^{a}_{\mu\nu} T^{\alpha}_{\rho a} + T^{a}_{\rho\mu} T^{\alpha}_{\nu a} + T^{a}_{\nu\rho} T^{\alpha}_{\mu a} := 0 \quad -(2)$$

Proof Consider $V^{\mu} V_{\mu} = V^{0} V_{0} + \ldots + V^{3} V_{3}$ $-(3)$

In eq. (3):

$$V^{a} = q^{a}_{\mu} V^{\mu}, \quad V_{a} = q^{\mu}_{a} V_{\mu} \quad -(4)$$

so $V^{a} V_{a} = q^{a}_{\mu} V^{\mu} q^{\mu}_{a} V_{\mu}$ $-(5)$

$$= q^{a}_{\mu} q^{\mu}_{a} V^{\mu} V_{\mu}$$

by the associative law of matrices. However

$$q^{a}_{\mu} q^{\mu}_{a} = 1 \quad -(6)$$

so $V^{a} V_{a} = V^{\mu} V_{\mu}$ $-(7)$

Q.E.D

So eq. (2) follows from eq. (1).

In matrix notation, restricting to two indices:

$$V^\mu = [V^1 \; V^2], \quad V_\mu = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad -(8)$$

so

$$V^\mu V_\mu = [V^1 \; V^2] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad -(9)$$

$$= V^1 V_1 + V^2 V_2$$

Denote:

$$q^\mu_{\ a} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad q^a_{\ \mu} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \quad -(10)$$

so

$$V^a = [V^1 \; V^2] \begin{bmatrix} E & F \\ G & H \end{bmatrix} \quad -(11)$$

$$= [V^1 E + V^2 G \quad V^1 F + V^2 H] \quad -(12)$$

$$V_a = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} AV_1 + BV_2 \\ CV_1 + DV_2 \end{bmatrix}$$

So

$$V^a V_a = [V^1 E + V^2 G \quad V^1 F + V^2 H] \times \begin{bmatrix} AV_1 + BV_2 \\ CV_1 + DV_2 \end{bmatrix} \quad -(13)$$

i.e.

$$\nabla^a \nabla_a = \nabla^1 \nabla_1 (AE + CF) + \nabla^2 \nabla_2 (BG + HD)$$
$$+ \nabla^2 \nabla_1 (AG + HC) + \nabla^1 \nabla_2 (EB + FD) \quad -(14)$$

Now use:

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} AE + CF & EB + FD \\ AG + CH & GB + HD \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(15)$$

so

$$AE + CF = BG + HD = 1 \quad -(16)$$
$$EB + FD = AG + HC = 0.$$

Fm eqs. (14) and (16):

$$\nabla^a \nabla_a = \nabla^1 \nabla_1 + \nabla^2 \nabla_2 = \nabla^\mu \nabla_\mu \quad -(17)$$

Q.E.D

Eq. (2) can be written as:

$$\left(T^a_{\mu\nu} T^b_{\rho a} + T^a_{\rho\mu} T^b_{\nu a} + T^a_{\nu\rho} T^b_{\mu a} \right) q^\alpha_b := 0 \quad -(18)$$

A possible solution of eq. (18) is

$$T^a_{\mu\nu} T^b_{\rho a} + T^a_{\rho\mu} T^b_{\nu a} + T^a_{\nu\rho} T^b_{\mu a} = 0 \quad -(19)$$

4) Proof

The structure of eq. (18) is:

$$\left(B^{b}_{\mu\nu\rho} + B^{b}_{\rho\mu\nu} + B^{b}_{\nu\rho\mu} \right) q^{\alpha}_{b} := 0 \quad -(20)$$

where

$$B^{b}_{\mu\nu\rho} := T^{a}_{\mu\nu} T^{b}_{\rho a} \quad -(21)$$

and so on. The simplest possible solution of eq. (20) is:

$$B^{b}_{\mu\nu\rho} + B^{b}_{\rho\mu\nu} + B^{b}_{\nu\rho\mu} = 0 \quad -(22)$$

i.e. for $b = (0), (1), (2), (3)$ $-(23)$

$$B^{(b)}_{123} + B^{(b)}_{312} + B^{(b)}_{231} = 0 \quad -(24)$$

Q.E.D

In electrodynamics the exact Evans identity is:

$$\left(F^{a}_{\mu\nu} F^{b}_{\rho a} + F^{a}_{\rho\mu} F^{b}_{\nu a} + F^{a}_{\nu\rho} F^{b}_{\mu a} \right) A^{\alpha}_{b} := 0, \quad -(25)$$

a particular solution of which is:

$$F^{a}_{\mu\nu} F^{b}_{\rho a} + F^{a}_{\rho\mu} F^{b}_{\nu a} + F^{a}_{\nu\rho} F^{b}_{\mu a} = 0 \quad -(26)$$