315(5): First and Second Bianchi Identities and Hodge Duality

The original first and second Bianchi identities are:

$$R^{\kappa}{}_{\mu\nu\rho} + R^{\kappa}{}_{\rho\mu\nu} + R^{\kappa}{}_{\nu\rho\mu} = 0 \quad -(1)$$

and

$$D_\rho R^{\kappa}{}_{\lambda\mu\nu} + D_\nu R^{\kappa}{}_{\lambda\rho\mu} + D_\mu R^{\kappa}{}_{\lambda\nu\rho} = 0 \quad -(2)$$

However the correct identities with torsion are:

$$R^{\kappa}{}_{\mu\nu\rho} + R^{\kappa}{}_{\rho\mu\nu} + R^{\kappa}{}_{\nu\rho\mu} := D_\mu T^{\kappa}{}_{\nu\rho} + D_\nu T^{\kappa}{}_{\rho\mu} + D_\rho T^{\kappa}{}_{\mu\nu} \quad -(3)$$

and

$$D_\rho R^{\kappa}{}_{\lambda\mu\nu} + D_\nu R^{\kappa}{}_{\lambda\rho\mu} + D_\mu R^{\kappa}{}_{\lambda\nu\rho} := T^{\alpha}{}_{\mu\nu} R^{\kappa}{}_{\lambda\rho\alpha} + T^{\alpha}{}_{\rho\mu} R^{\kappa}{}_{\lambda\nu\alpha} + T^{\alpha}{}_{\nu\rho} R^{\kappa}{}_{\lambda\mu\alpha} \quad -(4)$$

Eq. (3) is the Cartan identity and eq. (4) is the Jacobi Cartan Evans identity.

It is important to consider the Hodge duals of these identities, because they simplify and provide field equations. The Hodge dual of the torsion form for example is:

$$\tilde{T}^{a\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} T^{a}{}_{\alpha\beta} \quad -(5)$$

where $|g|$ is the determinant of the metric. It is also important to consider the identities (3) and (4) as identities of differential geometry:

a)

$$D_\mu T^a_{\nu\rho} + D_\nu T^a_{\rho\mu} + D_\rho T^a_{\mu\nu} := R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} \quad -(6)$$

and:

$$D_\rho R^a{}_{b\mu\nu} + D_\nu R^a{}_{b\rho\mu} + D_\mu R^a{}_{b\nu\rho} := T^c_{\mu\nu} R^a{}_{b\rho c} + T^c_{\rho\mu} R^a{}_{b\lambda\nu c} + T^c_{\nu\rho} R^a{}_{b\mu c} \quad -(7)$$

Here a, b and c are indices of the Cartan geometry. It seems clear that eqs. (6) and (7) are cyclic in μ, ν and ρ, in four dimensions.

The Hodge duals of eqs (6) and (7) are:

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^a{}_\mu{}^{\mu\nu} \quad -(8)$$

and:

$$D_\mu \tilde{R}^a{}_b{}^{\mu\nu} := T^c_{\mu\nu} \tilde{R}^a{}_b{}^\mu{}_c \quad -(9)$$

The structures of eqs. (8) and (9) are similar to the homogeneous field equation of the standard or $U(1)$ electrodynamics:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad -(10)$$

Eq (10) is equivalent to:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \quad -(11)$$

It is convenient to give details of this equivalence.

3) It follows from the Hodge dual transform in four dimensions in Minkowski spacetime:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \quad -(12)$$

in which the totally antisymmetric unit tensor in 4-D is:

$$\epsilon^{0123} = 1 = -\epsilon^{0213} = \epsilon^{0231} = \dots = 1 \quad -(13)$$

It follows that:

$$\left.\begin{aligned}
\tilde{F}^{03} &= F_{12} \\
\tilde{F}^{01} &= F_{23} \\
\tilde{F}^{02} &= F_{31} \\
\tilde{F}^{12} &= F_{03} \\
\tilde{F}^{13} &= F_{20} \\
\tilde{F}^{23} &= F_{01}
\end{aligned}\right] \quad -(14)$$

Eq. (10) gives four equations:

$$\partial_0\tilde{F}^{01} + \partial_2\tilde{F}^{21} + \partial_3\tilde{F}^{31} = 0 \quad -(15)$$

$$\partial_0\tilde{F}^{02} + \partial_1\tilde{F}^{12} + \partial_3\tilde{F}^{32} = 0 \quad -(16)$$

$$\partial_0\tilde{F}^{03} + \partial_1\tilde{F}^{13} + \partial_2\tilde{F}^{23} = 0 \quad -(17)$$

$$\partial_1\tilde{F}^{10} + \partial_2\tilde{F}^{20} + \partial_3\tilde{F}^{30} = 0 \quad -(18)$$

These are equivalent to:

$$\partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} = 0 \quad -(19)$$

$$\partial_0 F_{31} + \partial_1 F_{03} + \partial_3 F_{10} = 0 \quad -(20)$$

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \quad -(21)$$

$$\partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0 \quad -(22)$$

These are equivalent to:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(23)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(24)$$

using:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{bmatrix} \quad -(25)$$

and

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla} \right) \quad -(26)$$

So the same type of vector structure will emerge from eqs. (8) and (9). From previous work, eq. (8) is transformed into a field equation using:

$$\tilde{F}^{a\mu\nu} = A^{(0)} \tilde{T}^{a\mu\nu} \quad -(27)$$

and gives:

$$\underline{\nabla} \cdot \underline{B}^a = j^{0a} \quad -(27)$$

and

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{j}^a \quad -(28)$$

The vector structure of the magnetic charge current density:

5)

$$j^{a\mu} = \left(j^{0a}, \underline{j}^{a} \right) \quad -(29)$$

has been given in previous papers such as UFT 255.

So Eq. (9) will give new vector field equations.

As an example of this procedure consider eqs. (16) and (25), which give:

$$-\frac{1}{c}\frac{\partial}{\partial t}(cB_Y) + \frac{\partial E_Z}{\partial X} - \frac{\partial E_X}{\partial Z} = 0 \quad -(30)$$

i.e.

$$\frac{\partial B_Y}{\partial t} + \frac{\partial E_X}{\partial Z} - \frac{\partial E_Z}{\partial X} = 0. \quad -(31)$$

using the definition:

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ E_X & E_Y & E_Z \end{vmatrix} \quad -(32)$$

it becomes clear that eq. (31) is:

$$\frac{\partial B_Y}{\partial t} + (\underline{\nabla} \times \underline{E})_Y = 0 \quad -(33)$$

and is the Y component of

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(34)$$

So the same procedure can be applied to the Hodge dual (9) of the JCE identity to obtain its vector format. This will be done in the next note.