312(b): Evaluation of Energy Density due to Photon Rest Mass

This is defined as:

$$\frac{E}{V} = -\frac{\omega_0^2}{2c^3\pi^2}\int \frac{\hbar\omega}{e^y - 1}\, d\omega \quad - (1)$$

where

$$y = \hbar\omega / kT. \quad - (2)$$

It has been assumed that:

$$\omega_0 << \omega \quad - (3)$$

From eq. (2):

$$d\omega = \left(\frac{kT}{\hbar}\right) dy \quad - (4)$$

so

$$\frac{E}{V} = -\frac{\hbar\omega_0^2}{2c^3\pi^2}\left(\frac{kT}{\hbar}\right)^2 \int \frac{y\,dy}{e^y - 1} \quad - (5)$$

The definite integral:

$$\int_0^\infty \frac{y}{e^y - 1}\, dy = \frac{(2\pi)^2 B_1}{4} = \frac{\pi^2}{6} \quad - (6)$$

is known, but the lower bound in the integral (5) must be ω_0, not zero, so the Stefan Boltzmann law due to photon mass is:

$$\boxed{\frac{E}{V} = -\frac{\hbar\omega_0^2}{2c^3\pi^2}\left(\frac{kT}{\hbar}\right)^2 \int_{\omega_1}^\infty \frac{y\,dy}{e^y - 1}} \quad - (7)$$

2) where
$$\omega_1 >> \omega_0 \quad -(8)$$
It is necessary to evaluate the integral:
$$I = \int \frac{y\,dy}{e^y - 1} \quad -(9)$$
The Wolfram integrator gives:
$$I = Li_2 - \frac{y^2}{2} + y \log_e(1 - e^y) \quad -(10)$$
where
$$Li_n(y) = \sum_{k=1}^{\infty} \frac{y^k}{k^n} \quad -(11)$$
is the polylogarithmic function. However eq. (10) seems to be incorrect because for $y > 0$ it contains the logarithm of a negative number. Therefore it is necessary to work out the integral (9) with Maxima.

The rigorously correct expression for the density of states is:
$$\frac{dN}{V} = \frac{1}{3c^3\pi^2}\left(\left((\omega + d\omega)^2 - \omega_0^2\right)^{3/2} - \left(\omega^2 - \omega_0^2\right)^{3/2}\right) \quad -(12)$$

If the photon mass is neglected:
$$\omega_0 \rightarrow 0 \quad -(13)$$
then eq. (12) reduces to the equation used by

) Rayleigh:

$$\frac{dN}{V} = \frac{1}{3c^3\pi^2}\left((\omega + d\omega)^3 - \omega^3\right) \quad -(14)$$

$$= \frac{1}{3c^3\pi^2}\left(\omega^3 + 3\omega^2 d\omega + 3\omega(d\omega)^2 + (d\omega)^3 - \omega^3\right)$$

Rayleigh neglected higher order infinitesimal to arrive at:

$$\frac{dN}{V} = \frac{\omega^2}{c^3\pi^2}\, d\omega \quad -(15)$$

So

$$\frac{dE}{V} = \frac{\omega^2 \langle \hbar\omega \rangle}{c^3\pi^2}\, d\omega$$

$$= \frac{\hbar\omega^3\, d\omega}{c^3\pi^2(e^y - 1)} \quad -(16)$$

The Stefan Boltzmann law is then:

$$\frac{E}{V} = \int_0^\infty \frac{\hbar\omega^3\, d\omega}{c^3\pi^2(e^y - 1)}$$

$$= \left(\frac{\pi^2 k^4}{15c^3\hbar^3}\right) T^4 \quad -(17)$$

As in note 310(5), eq. (12) the presence of photon rest mass changes eq. (15) to:

$$\frac{dN}{V} = \frac{\Omega^2}{c^3\pi^2}\, d\Omega \quad -(18)$$

where
$$\Omega^2 = \omega^2 - \omega_0^2 \quad -(19)$$

Using:
$$\frac{d\Omega}{d\omega} = \frac{\omega}{(\omega^2-\omega_0^2)^{1/2}} \quad -(20)$$

then:
$$\frac{dN}{V} = \frac{\omega}{c^3\pi^2}\left(\omega^2-\omega_0^2\right)^{1/2} d\omega \quad -(21)$$

So:
$$\frac{E}{V} = \frac{\hbar}{c^3\pi^2}\int \frac{\omega^2\left(\omega^2-\omega_0^2\right)^{1/2}}{e^{y}-1}\, d\omega \quad -(22)$$

The equivalent expression for monochromatic radiation is:
$$\Phi = c\frac{E}{V} = \frac{\hbar\left(\omega^2-\omega_0^2\right)^2}{3c^2\pi^2\left(e^{y}-1\right)} \quad -(23)$$

so to avoid complications it is much easier to use monochromatic radiation.

In the approximation:
$$\hbar\omega \ll kT \quad -(24)$$

eq. (23) reduces to:
$$\Phi = \frac{kT}{3c^2}\frac{\left(\omega^2-\omega_0^2\right)^2}{\omega} \quad -(25)$$

5) using the Beer Lambert law:

$$\frac{\Phi}{\Phi_i} = \exp(-\alpha \ell) \quad -(26)$$

Then:

$$\left(\frac{\omega^2 - \omega_0^2}{\omega_i^2 - \omega_0^2}\right)^2 \frac{\omega_i}{\omega} = \exp(-\alpha \ell) \quad -(27)$$

So ω_0 can be found from two measurable frequencies ω_i and ω, and the power absorption coefficient α.

From eq. (27):

$$\frac{\omega^2 - \omega_0^2}{\omega_i^2 - \omega_0^2} = \left(\frac{\omega}{\omega_i}\right)^{1/2} \exp\left(-\frac{\alpha \ell}{2}\right) \quad -(28)$$

$$:= A$$

So:

$$\omega^2 - \omega_0^2 = A\left(\omega_i^2 - \omega_0^2\right) \quad -(29)$$

i.e.

$$\omega_0^2 (1 - A) = \omega^2 - A\omega_i^2 \quad -(30)$$

so

$$\boxed{\omega_0^2 = \frac{\omega^2 - A\omega_i^2}{1 - A}} \quad -(31)$$