312(5): The Density of States due to Photon Mass

In previous work the density of states was defined by:

$$\frac{dN}{V} = \frac{1}{3c^3\pi^2}\left(\left((\omega+d\omega)^2-\omega_0^2\right)^{3/2}-\left(\omega^2-\omega_0^2\right)^{3/2}\right) \quad -(1)$$

In these expressions:

$$\left(\omega^2-\omega_0^2\right)^{3/2} = \left(\omega^2\left(1-\frac{\omega_0^2}{\omega^2}\right)\right)^{3/2}$$

$$= \omega^3\left(1-\frac{\omega_0^2}{\omega^2}\right)^{3/2} \quad -(2)$$

If

$$\omega_0 \ll \omega \quad -(3)$$

then

$$\left(1-\frac{\omega_0^2}{\omega^2}\right)^{3/2} \sim 1-\frac{3}{2}\left(\frac{\omega_0}{\omega}\right)^2 \quad -(4)$$

Similarly:

$$\left((\omega+d\omega)^2-\omega_0^2\right)^{3/2} = (\omega+d\omega)^3\left(1-\frac{\omega_0^2}{(\omega+d\omega)^2}\right)^{3/2}$$

$$\sim (\omega+d\omega)^3\left(1-\frac{3}{2}\frac{\omega_0^2}{(\omega+d\omega)^2}\right)$$

$$= (\omega+d\omega)^3-\frac{3}{2}\omega_0^2(\omega+d\omega) \quad -(5)$$

Therefore:

$$\frac{dN}{V} \sim \frac{1}{3c^3\pi^2}\left((\omega+d\omega)^3-\frac{3}{2}\omega_0^2(\omega+d\omega) - \omega^3\left(1-\frac{3}{2}\left(\frac{\omega_0}{\omega}\right)^2\right)\right) \quad -(6)$$

2)

$$= \frac{1}{3c^3\pi^2}\left((\omega + d\omega)^3 - \omega^3 - \frac{3}{2}\omega_0^2(\omega + d\omega) + \frac{3}{2}\omega^3\left(\frac{\omega_0}{\omega}\right)^2 \right) \quad -(7)$$

$$= \frac{1}{3c^3\pi^2}\left((\omega + d\omega)^3 - \omega^3 + \frac{3}{2}\left(\omega\,\omega_0^2 - \omega_0^2\,\omega - \omega_0^2\,d\omega\right) \right)$$

Therefore:

$$\frac{dN}{V} = \frac{1}{3c^3\pi^2}\left((\omega + d\omega)^3 - \omega^3\right) - \frac{\omega_0^2}{2c^3\pi^2}\,d\omega \quad -(8)$$

The first term on the RHS of eq. (8) is the usual Rayleigh Jeans density of states but the photon rest mass m_0 contributes the term:

$$\boxed{\frac{dN}{V} = -\frac{\omega_0^2}{2c^3\pi^2}\,d\omega} \quad -(9)$$

where the photon rest frequency is:

$$\omega_0 = m_0 c^2/\hbar \quad -(10)$$

the energy density due to eq. (9) is:

$$\frac{dE}{V} = -\frac{\omega_0^2}{2c^3\pi^2}\langle \hbar\omega \rangle\,d\omega \quad -(11)$$

where

$$\langle \hbar\omega \rangle = \frac{\hbar\omega}{e^y - 1} \quad -(12)$$

with

$$y = \frac{\hbar\omega}{kT} \quad -(13)$$

3). In the high temperature approximation:

$$\hbar\omega \ll kT \quad -(14)$$

then

$$\frac{dE}{V} = -\frac{\omega_0^2 kT}{2c^3\pi^2}\, d\omega. \quad -(15)$$

Therefore the correction to the Rayleigh Jeans law due to photon rest mass leads to the energy density correction:

$$\frac{E}{V} = -\left(\frac{\omega_0^2 kT}{2c^3\pi^2}\right)\omega \quad -(16)$$

by integrating eq. (15). Therefore:

$$\boxed{\Phi = c\frac{E}{V} = -\left(\frac{\omega_0^2 kT}{2c^2\pi^2}\right)\omega \quad \text{for } \hbar\omega \ll kT.} \quad -(17)$$

For black body radiation:

$$\Phi = -\left(\frac{\omega_0^2 kT}{2c^2\pi^2}\right)\int_0^\infty d\omega \quad -(18)$$

$$\longrightarrow \infty$$

Therefore the photon rest mass affects the Stefan Boltzmann law very considerably if high enough frequencies are taken into account.

4) The rigorous calculation of the correction is:

$$\bar{\Phi} = \frac{c\bar{F}}{\bar{V}} = \frac{-\omega_0^2}{2c^2\pi^2} \int \frac{\hbar\omega}{e^y - 1} d\omega \quad -(19)$$

and for black body radiation:

$$\bar{\Phi} = -\frac{\hbar\omega_0^2}{2c^2\pi^2} \int_0^\infty \frac{\omega}{e^y - 1} d\omega \quad -(20)$$

The integral can be worked out with:

$$\int_0^\infty \left(\frac{x^{2n-1}}{e^x - 1} \right) dx = (2\pi)^{2n} \frac{B_n}{4n} \quad -(21)$$

where $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, ...

are Bessel functions

The important thing to note is that the correction due to photon rest mass of the Stefan Boltzmann law becomes very large when:

$$\hbar\omega << kT \quad -(22)$$

and at very high frequencies ω. Otherwise from eq. (20) it is very small.

For example, if two gamma ray frequencies ω_1 and ω_2 define a range of γ ray frequencies then from eq. (7):

$$\bar{\Phi} = -\left(\frac{\omega_0^2 kT}{2c^3\pi^2}\right)(\omega_1 - \omega_2) \quad -(23)$$

in which:

$$k = 1.38066 \times 10^{-23}\ JK^{-1} \quad -(24)$$

and

$$c = 2.997925 \times 10^{8}\ ms^{-1} \quad -(25)$$