292(2): Conservation of Polychromatic Intensity in Refraction and Reflection.

It is assumed that the incident intensity is I, and that the refracted and reflected intensities are I_1 and I_2 respectively, so:

$$I = I_1 + I_2 \quad -(1)$$

The calculation of polychromatic intensity in the original Rayleigh Jeans theory and Planck theory is based on the density of states:

$$\frac{dN}{V} = \frac{\omega^2}{\pi^2 c^3} d\omega = \frac{8\pi}{c^3} f \, df \quad -(2)$$

where

$$\omega = 2\pi f. \quad -(3)$$

However notes 291 and UFT291 show that eq. (2) must be corrected to:

$$\frac{dN}{V} = \frac{\omega^2}{\pi^2 c^3} d\omega + \frac{\omega}{\pi^2 c^3} (d\omega)^2 + \frac{(d\omega)^3}{3\pi^2 c^3}$$

$$= \frac{10}{3} \frac{\omega^2}{\pi^2 c^3} d\omega \quad -(4)$$

The energy density infinitesimal is:

$$\frac{dU}{V} = \langle E \rangle \frac{dN}{V} = \hbar\omega \left(\frac{x}{1-x} \right) \frac{\omega^2}{\pi^2 c^3} d\omega$$

$$= \frac{\hbar \omega^3}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega \quad -(5)$$

x) This is converted to:

$$\frac{dU}{V} = \frac{10\hbar\omega^3}{3\pi^2c^3}\left(\frac{x}{1-x}\right)d\omega \quad -(6)$$

i.e.

$$\frac{U}{V} = \frac{10\hbar}{3\pi^2c^3}\int\omega^3\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^{-1}d\omega \quad -(7)$$

The polychromatic intensity in watts per square metre is:

$$I = \frac{cU}{V} = \frac{10\hbar}{3\pi^2c^2}\int\omega^3\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^{-1}d\omega \quad -(8)$$

The refracted intensity is:

$$I_1 = \frac{10\hbar}{3\pi^2c^2}\int\omega_1^3\left(\exp\left(\frac{\hbar\omega_1}{kT}\right)-1\right)^{-1}d\omega_1 \quad -(9)$$

and the reflected intensity is:

$$I_2 = \frac{10\hbar}{3\pi^2c^2}\int\omega_2^3\left(\exp\left(\frac{\hbar\omega_2}{kT}-1\right)\right)^{-1}d\omega_2 \quad -(10)$$

The correction factor of 10/3 is a large correction to the derivation of the Stefan Boltzmann law:

3)

$$I = \frac{10\hbar}{3\pi^2 c^2} \int_0^\infty \frac{\omega^3 d\omega}{(e^x - 1)} \quad -(11)$$

where

$$x = \frac{\hbar\omega}{kT} \quad -(12)$$

So

$$I = \frac{10}{3} \cdot \frac{\pi^2}{15} \left(\frac{k^4}{c^2 \hbar^3} \right) T^4 \quad -(13)$$

The original result is :

$$I(\text{Rayleigh Jeans}) = \frac{\pi^2}{15} \left(\frac{k^4}{c^2 \hbar^3} \right) T^4 \quad -(14)$$

based on the average energy of the Planck oscillator:

$$\langle E \rangle = \frac{\hbar\omega}{e^x - 1} \quad -(15)$$

provided that:

$$\hbar\omega < kT. \quad -(16)$$

However from eqs. (1), (8), (9) and (10) :

$$\int \frac{\omega^3 d\omega}{e^x - 1} = \int \frac{\omega_1^3 d\omega_1}{e^{x_1} - 1} + \int \frac{\omega_2^3 d\omega_2}{e^{x_2} - 1} \quad -(17)$$

where

$$x_1 = \hbar\omega_1 / (kT_1) \quad -(18)$$

$$x_2 = \hbar\omega_2 / (kT_2) \quad -(19)$$

so the correction factor of 10/3 does not

4) after eq. (17). In general it is assumed that the temperature of the three beams are T, T_1 and T_2. Therefore in the process of reflection and refraction of a polychromatic beam the distribution of frequencies of the incident beam gives rise to distributions of frequencies of the refracted and reflected beams.

If it is assumed that the frequencies of the incident beam are distributed from ω_1 to ω_2, then:

$$\underline{I} = \int_{\omega_1}^{\omega_2} \frac{\omega^3 \, d\omega}{e^{x} - 1} \quad -(21)$$

where

$$x = \frac{\hbar \omega}{kT} \quad -(22)$$

Similarly:

$$I_1 = \int_{\omega_3}^{\omega_4} \frac{\omega_1^3 \, d\omega_1}{e^{x_1} - 1} \quad -(23)$$

and

$$I_2 = \int_{\omega_5}^{\omega_6} \frac{\omega_2^3 \, d\omega_2}{e^{x_2} - 1} \quad -(24)$$

These integrals must be worked out with Maxima or numerically. So:

5)
$$\int_{\omega_1}^{\omega_2} \frac{\omega^3 d\omega}{e^x - 1} = \int_{\omega_3}^{\omega_4} \frac{\omega_1^3 d\omega_1}{e^{x_1} - 1} + \int_{\omega_5}^{\omega_6} \frac{\omega_2^3 d\omega_2}{e^{x_2} - 1} \quad -(25)$$

in general.

In order to simplify the calculation assume that:

$$x << 1, \; x_1 << 1, \; x_2 << 1 \quad -(26)$$

so

$$e^x \sim 1 + x \quad -(27)$$

$$e^{x_1} \sim 1 + x_1 \quad -(28)$$

$$e^{x_2} \sim 1 + x_2 \quad -(29)$$

Then

$$\underline{I} \sim \int_{\omega_1}^{\omega_2} \left(\frac{kT}{\hbar\omega} \right) \omega^3 d\omega \quad -(30)$$

$$= \frac{kT}{3\hbar} \left(\omega_2^3 - \omega_1^3 \right) \quad -(31)$$

Eq. (25) reduces to: $\quad -(32)$

$$\boxed{T\left(\omega_2^3 - \omega_1^3\right) = T_1\left(\omega_4^3 - \omega_3^3\right) + T_2\left(\omega_6^3 - \omega_5^3\right)}$$

So in general there will be a distribution of refracted and reflected frequencies.

b)

The Beer Lambert law for an absorbing and refracting medium means that:

$$I_1 = I \exp(-\alpha r) \quad -(33)$$

where α is the power absorption coefficient. If it is assumed that:

$$T = T_1 = T_2 \quad -(34)$$

then in the approximations (26) to (32):

$$(\omega_4^3 - \omega_3^3) = (\omega_2^3 - \omega_1^3)\, e^{-\alpha r} \quad -(35)$$

so

$$(\omega_4^3 - \omega_3^3) < (\omega_2^3 - \omega_1^3) \quad -(36).$$

Therefore the range of frequencies is narrower in an absorbing and refracting medium. This can be used as an experimental investigation of the Evans / Morris effects.

In total internal reflection:

$$\omega_2^3 - \omega_1^3 = \omega_6^3 - \omega_5^3 \quad -(37)$$

so the range of frequencies is the same in total internal reflection. This is another test of the Evans Morris effects.

At the Brewster angle:

$$\omega_2^3 - \omega_1^3 = \omega_4^3 - \omega_3^3 \quad -(38)$$

and the range is again the same.