Note 292(1): Combination of Conservation of Intensity, Beer Lambert Law and Snell's Law.

Consider an incident beam of intensity I, a refracted beam of intensity I_1 and a reflected beam of intensity I_2. Then:

$$I = I_1 + I_2 \quad -(1)$$

For a monochromatic beam of incident frequency ω:

$$I = \left(\frac{x}{1-x}\right)\frac{\hbar\omega^4}{6\pi^2c^2} \quad -(2)$$

$$= c\langle E\rangle\frac{N}{V}$$

Here N/V is the number of oscillators in a volume V:

$$\frac{N}{V} = \frac{1}{6\pi^2}\left(\frac{\omega}{c}\right)^3 \quad -(3)$$

of the 1900 Rayleigh theory, and

$$\langle E\rangle = \left(\frac{x}{1-x}\right)\hbar\omega \quad -(4)$$

is the average energy of a Planck oscillator.

In the refracting medium the beam is absorbed according to the Beer Lambert law:

$$I_1 = I\exp\left(-\alpha r\right) \quad -(5)$$

1) and Snell's laws demand that:

$$\theta = \theta_2 \quad - (6)$$

and

$$n \sin\theta = n_1 \sin\theta_1 \quad - (7)$$

In eq. (5) α is the power absorption coefficient and r is the distance travelled into the absorbing and refracting medium. The power absorption coefficient is:

$$\alpha = \frac{\omega_1 \epsilon_1''}{c n_1'} \quad - (8)$$

where ϵ_1'' is the dielectric loss of the medium and n_1' is its refractive index:

$$n_1'^2 = \frac{1}{2}\left(\epsilon_1' + \left(\epsilon_1'^2 + \epsilon_1''^2\right)^{1/2}\right) - (9)$$

If it is assumed that the incident medium is air then:

$$n_1'^2 = \left(\frac{\sin\theta}{\sin\theta_1}\right)^2 - (10)$$

The refracted intensity is:

$$I_1 = \left(\frac{x_1}{1-x_1}\right)\frac{L\omega_1^4}{6\pi^2 c^2} - (11)$$

So:

$$\left(\frac{x_1}{1-x_1}\right)\omega_1^4 = \left(\frac{x}{1-x}\right)\omega^4 \exp\left(-\alpha r\right) - (12)$$

3) i. e.

$$\left(\frac{1}{\exp\left(\frac{\hbar\omega_1}{kT}\right)-1}\right)\omega_1^4 = \left(\frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right)-1}\right)\omega^4 \exp\left(-\alpha r\right) \quad -(13)$$

If $\hbar\omega_1 \ll kT$ — (14)

and $\hbar\omega \ll kT$ — (15)

then

$$\boxed{\left(\frac{\omega_1}{\omega}\right)^3 = \exp\left(-\alpha Z\right)} \quad -(16)$$

The refracted frequency is shifted to the red:

$$\omega_1 < \omega. \quad -(17)$$

Use the Debye model for the refracting medium, for example water:

$$\epsilon_1' = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1+\omega_1^2\tau^2} \quad -(18)$$

and

$$\epsilon_1'' = (\epsilon_0 - \epsilon_\infty)\frac{\omega_1\tau}{1+\omega_1^2\tau^2}. \quad -(19)$$

The incident frequency can be calculated as follows from the refracted frequency:

4)
$$\omega^3 = \omega_1^3 \exp(\alpha r) \quad -(20)$$
where:
$$\alpha = \frac{\omega_1 \epsilon_1''(\omega_1)}{c\, n_1(\omega_1)} \quad -(21)$$
and
$$n_1'^2 = \frac{1}{2}\left(\epsilon_1' + \left(\epsilon_1'^2 + \epsilon_1''^2\right)^{1/2}\right) - (22)$$
with
$$\epsilon_1' = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1 + \omega_1^2 \tau^2} \quad -(23)$$
and
$$\epsilon_1'' = (\epsilon_0 - \epsilon_\infty)\frac{\omega_1 \tau}{1 + \omega_1^2 \tau^2} \quad -(24).$$

So for a given r and τ, ϵ_0 and ϵ_∞, ω can be found from ω_1.

In general, for any experimentally measured $\epsilon_1''(\omega_1)$ and $n_1(\omega_1)$ of the refracting medium, or for any experimentally measured power absorption coefficient $\alpha(\omega_1)$, ω can be found for a given ω_1.

For a given spectrum or power absorption coefficient:
$$\alpha = \frac{1}{r} \log_e \frac{I}{I_1} \quad -(25)$$

i.e.

$$\alpha(\omega_1) = \frac{1}{r}\log_e\left(\left(\frac{\omega}{\omega_1}\right)^3\right) \quad -(26)$$

Therefore there is a range of ω_1 generated by a single incident ω.

Suggested Procedure

Using eq. (20) calculate ω for a given ω_1 and r using the Debye model. Plot ω as a function of ω_1 and r, and find the combination of ω_1 and r that keeps ω constant. The problem is to find r so that

$$\omega^3 = \text{constant} = \omega_1^3 \exp\left(r\alpha(\omega_1)\right) \quad -(27)$$

therefore keep ω the same for a range of ω_1 and r.

Similarly for eqs (9) and (10) there is a range of θ_1 for a given θ:

$$\sin^2\theta_1 = \frac{\sin^2\theta}{n_1'^2} \quad -(27)$$

$$= \frac{2\sin^2\theta}{\left(\epsilon_1' + \left(\epsilon_1'^2 + \epsilon_1''^2\right)^{1/2}\right)}$$

In general $n_1'^2$ is a function of ω_1 so:

6)

$$n_1'^2 = f(\omega_1) \quad -(28)$$

and

$$\sin^2\theta_1 = \frac{\sin^2\theta}{f(\omega_1)} \quad -(29)$$

If there is a fixed angle of refraction and reflection then n_1' is a constant:

$$n_1' = \frac{\sin\theta}{\sin\theta_1} \quad -(30)$$

In the case of a low absorbing glass it can be assumed that $d\alpha(\omega_1)$ is approximately a constant, so:

$$\alpha(\omega_1) \sim \alpha_0 \quad -(31)$$

and

$$\omega_1^3 = \omega^3 \exp(-\alpha_0 r) \quad -(32)$$

Finally the reflected intensity is found from:

$$I = I \exp(-\alpha r) + I_2 \quad -(33)$$

i.e

$$I_2 = I(1 - \exp(-\alpha r)) \quad -(34)$$

$$\frac{I_2}{I} \sim \left(\frac{\omega_2}{\omega}\right)^3 = 1 - \exp(-\alpha r) \quad -(35)$$