291(2): A Note on the Riemann Integral Method.

In this note the meaning of the integration:

$$I = \int (dx)^{1/2} \quad -(1)$$

is investigated with the Riemann integral method.

Consider a function:

$$y = f(x) \quad -(2)$$

which is positive and continuous in the range $a \leq x \leq b$:

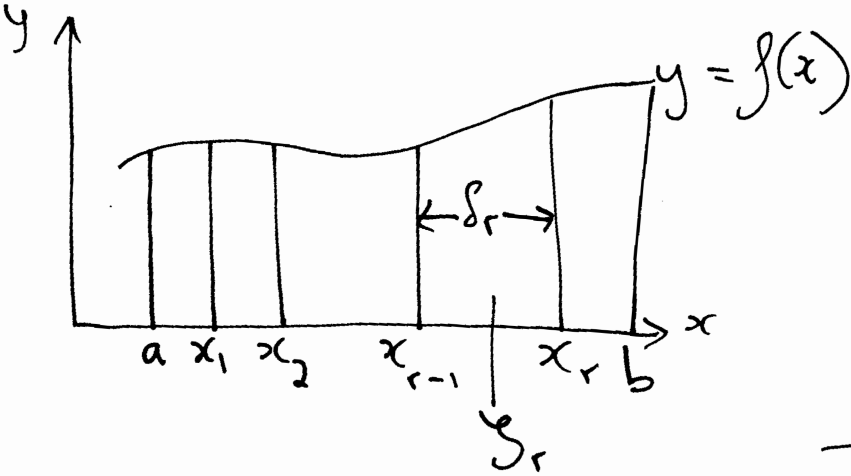

Fig. (1)

$$-(3)$$

Here:

$$a (= x_0) < x_1 < x_2 < \ldots < x_{n-2} < x_{n-1} < b (= x_n)$$

and

$$x_{r-1} < \xi_r < x_r \quad -(4)$$

with

$$\delta_r = x_r - x_{r-1}. \quad -(5)$$

The area between x_{r-1} and x_r is approximated by:

$$A = f(\xi_r)\, \delta_r \quad -(6)$$

The area bounded by the curve, the x axis and the lines $x = a$ and $x = b$ is the sum:

2)

$$S_n = \sum_{r=1}^{n} f(\xi_r)\,\delta_r \quad -(7)$$

The definite integral:

$$I = \int_a^b f(x)\,dx \quad -(8)$$

is defined by $n \to \infty$ and $\delta_r \to 0$ in eq. (7):

$$I = \lim_{\substack{n \to \infty \\ \delta_r \to 0}} \sum_{r=1}^{n} f(\xi_r)\,\delta_r \quad -(9)$$

If

$$f(\xi_r) = 1 \quad -(10)$$

then:

$$I = \lim_{\substack{n \to \infty \\ \delta_r \to 0}} \sum_{r=1}^{n} \delta_r \quad -(11)$$

where

$$\delta_r = x_r - x_{r-1} \quad -(12)$$

so

$$I = \lim_{\delta_r \to 0} \sum_{r=1}^{\infty} (x_r - x_{r-1})$$

$$= \int_a^b dx \quad -(13)$$

$$= b - a$$

Now consider the integral:

3)

$$I = \int_a^b (dx)^{1/2} \quad -(14)$$

This type of integral is used in the coverin to the Rayleigh Jeans law given in Note 290(1). The integral (14) is defined by:

$$I = \lim_{\delta_r \to 0} \sum_{r=1}^{\infty} (x_r - x_{r-1})^{1/2} \quad -(15)$$

We wish to investigate under what circumstances the following is true:

$$\left(\int_a^b dx \right)^{1/2} = \int_a^b (dx)^{1/2} \quad -(16)$$

Denote: $\delta_r = y_r = x_r - x_{r-1}$ $-(17)$

then eq (16) means that:

$$y_1^{1/2} + y_2^{1/2} + \dots + y_n^{1/2} = (y_1 + y_2 + y_3 + \dots + y_n)^{1/2}$$

$-(18)$

Squaring both sides:

$$y_1 + y_2 + \dots + y_n = (y_1^{1/2} + y_2^{1/2} + \dots + y_n^{1/2})^2 \quad -(19)$$

i.e. $\delta_1 + \delta_2 + \dots + \delta_n = (\delta_1^{1/2} + \delta_2^{1/2} + \dots + \delta_n^{1/2})^2$ $-(20)$

4) Considering the first two terms:

$$\left(\delta_1^{1/2} + \delta_2^{1/2}\right)^2 = \delta_1 + \delta_2 + 2\delta_1^{1/2}\delta_2^{1/2} \quad -(21)$$

$$\rightarrow \delta_1 + \delta_2 \quad -(22)$$

if $\delta_1 \rightarrow 0$ and $\delta_2 \rightarrow 0$

to a very rough approximation. $\quad -(23)$

Considering the first three terms:

$$\left(\delta_1^{1/2} + \delta_2^{1/2} + \delta_3^{1/2}\right)^2$$

$$= \delta_1 + \delta_2 + \delta_3 + 2\left(\delta_1^{1/2}\delta_2^{1/2} + \delta_3^{1/2}\left(\delta_1^{1/2} + \delta_2^{1/2}\right)\right)$$

so:

$$\left(\delta_1^{1/2} + \delta_2^{1/2} + \delta_3^{1/2} + \dots + \delta_n^{1/2}\right)^2$$

$$= \delta_1 + \delta_2 + \delta_3 + \dots + \delta_n$$

$$+ 2\left(\delta_2^{1/2}\delta_1^{1/2} + \delta_3^{1/2}\left(\delta_1^{1/2} + \delta_2^{1/2}\right)\right. \quad -(24)$$

$$+ \delta_4^{1/2}\left(\delta_1^{1/2} + \delta_2^{1/2} + \delta_3^{1/2}\right)$$

$$\left. + \dots + \delta_{n+1}^{1/2}\left(\delta_1^{1/2} + \delta_2^{1/2} + \dots + \delta_n^{1/2}\right)\right)$$

where

$$\left(\int (dx)^{1/2}\right)^2 = \sum_{\substack{r=1 \\ \delta_r \rightarrow 0}}^{\infty} \delta_r^{1/2} \quad -(25)$$

and:

$$\int dx = \sum_{\substack{r=1 \\ \delta_r \to 0}}^{\infty} \delta_r \quad - (26)$$

From eq. (24) it is seen that:

$$\left(\int_a^b (dx)^{1/2} \right)^2 \xrightarrow[\delta_r \to 0]{} \int_a^b dx \quad - (27)$$

This is a useful result that can be used in note 290(1).

Similarly:

$$\left(\int_a^b (dx)^{1/n} \right)^n \xrightarrow[\delta_r \to 0]{} \int_a^b dx \quad - (28)$$

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5) and:

$$\int dx = \sum_{\substack{r=1 \\ \delta_r \to 0}}^{\infty} \delta_r \quad - (26)$$

From eq. (24) it is seen that:

$$\left(\int_a^b (dx)^{1/2} \right)^2 \xrightarrow[\delta_r \to 0]{} \int_a^b dx \quad - (27)$$

This is a useful result that can be used in note 290 (1).

Similarly:

$$\left(\int_a^b (dx)^{1/n} \right)^n \xrightarrow[\delta_r \to 0]{} \int_a^b dx \quad - (28)$$
