291(1): Correction of the Rayleigh Jeans Density of States for Higher Order Infinitesimals

The correct Rayleigh Jeans density of states, from elementary considerations, is:

$$\frac{dN}{V} = \frac{\omega^2}{\pi^2 c^3} d\omega + \frac{\omega}{\pi^2 c^3}(d\omega)^2 + \frac{(d\omega)^3}{3\pi^2 c^3} \quad -(1)$$

This correction is a large one, and is due to the second two terms in eq. (1). These were left out both by Rayleigh in 1900 and by Jeans in 1905.

Eq. (1) is derived from the number density of oscillators of the original Rayleigh calculation:

$$\frac{N}{V} = \frac{\omega^3}{6c^3\pi^2} \quad -(2)$$

so the infinitesimal of number density is:

$$\frac{dN}{V} = \frac{1}{6c^3\pi^2}\left((\omega+d\omega)^3 - \omega^3\right) \quad -(3)$$

$$= \frac{1}{6c^3\pi^2}\left(\left(\omega^2 + 2\omega d\omega + (d\omega)^2\right)\left(\omega + d\omega\right) - \omega^3\right)$$

$$= \frac{1}{6c^3\pi^2}\Big(\omega^3 + 2\omega^2 d\omega + \omega(d\omega)^2 + \omega^2 d\omega$$
$$+ 2\omega(d\omega)^2 + (d\omega)^3 - \omega^3\Big)$$

2)

$$= \frac{1}{6c^3\pi^2}\left(3\omega^2 d\omega + 3\omega(d\omega)^2 + (d\omega)^3\right)$$

$$= \frac{\omega^2 d\omega}{2c^3\pi^2} + \frac{\omega(d\omega)^2}{2c^3\pi^2} + \frac{(d\omega)^3}{6c^3\pi^2}$$

Rayleigh doubled the result by assuming that there are two states of polarization of each radiation oscillator, giving eq. (1), QED.

The number of oscillators per unit volume is found from eq. (1) by integrating over all frequencies in eq. (1). Rayleigh left out the second two terms in eq. (1) so in this calculation:

$$\frac{N}{V} = \int \frac{\omega^2}{\pi^2 c^3}\, d\omega = \frac{\omega^3}{3\pi^2 c^3} \quad -(4)$$

This result is incorrect at high frequency because it goes to infinity. It was therefore corrected by the Planck quantum theory, in which the average energy of an oscillator is:

$$\langle E \rangle = \left(\frac{x}{1-x}\right)\hbar\omega \quad -(5)$$

where

$$x = \exp\left(-\frac{\hbar\omega}{kT}\right) \quad -(6)$$

3) The infinitesimal of energy density from this theory is:

$$\frac{d\bar{U}}{\bar{V}} = \langle E \rangle \frac{dN}{\bar{V}} \quad -(7)$$

and the energy density is found by integrating over frequency:

$$\frac{\bar{U}}{V} = \int \frac{d\bar{U}}{\bar{V}} = \int \langle E \rangle \frac{dN}{\bar{V}} \, d\omega \quad -(8)$$

The old theory relied on the assumption that:

$$\frac{dN}{\bar{V}} \overset{?}{=} \frac{\omega^2}{\pi^2 c^3} \, d\omega \quad -(9)$$

so:

$$\frac{\bar{U}}{\bar{V}} = \int \frac{\hbar \omega^3}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega \quad -(10)$$

For black body radiation:

$$\frac{\bar{U}}{\bar{V}} = \int_0^\infty \frac{\hbar \omega^3}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega$$

$$= \left(\frac{\pi^2}{15} \frac{k^4}{c^3 \hbar^3} \right) T^4 \quad -(11)$$

This is the old Stefan Boltzmann law, derived from the Planck quantum theory. The energy per volume

4) of radiation in a black body consisting of all possible frequencies is proportional to the fourth power of temperature.

The number of photons of energy $\hbar\omega$ per volume V is, from eq. (4):

$$\frac{N_p}{V} = \int \frac{\langle E \rangle}{\hbar\omega} \frac{dN}{V} \, d\omega \quad -(12)$$

$$= \int \frac{\omega^2}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega$$

For black body radiation:

$$\frac{N_p}{V} = \int_0^\infty \frac{\omega^2}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega$$

$$= \left(\frac{2\zeta(3)}{\pi^2} \left(\frac{k}{c\hbar} \right)^3 \right) T^3 \quad -(13)$$

where

$$\zeta(3) = 1.20206 \quad -(14)$$

is the third zeta function defined by:

$$\int_0^\infty \left(\frac{x^{2n}}{e^x - 1} \right) dx = (2n)! \, \zeta(2n+1) \quad -(15)$$

In the old theory the number of photons of

5) energy E_ω in a volume V of black body radiation is proportional to the cube of temperature.

Both calculations depend on the Rayleigh Jeans density of states, dN / V.

The correct density of states, however, is eq. (1), in which the second two terms contribute more oscillators per unit volume as follows.

Write eq. (1) as:

$$\frac{dN}{V} = \frac{1}{V}\left(dN_1 + dN_2 + dN_3\right) \quad - (16)$$

where:

$$\frac{dN_2}{V} = \frac{\omega}{\pi^2 c^3}(d\omega)^2 \quad - (17)$$

and

$$\frac{dN_3}{V} = \frac{(d\omega)^3}{3\pi^2 c^3} \quad - (18)$$

with the Rayleigh density of states:

$$\frac{dN_1}{V} = \frac{\omega^2}{\pi^2 c^3} d\omega \quad - (19)$$

From eq. (17):

$$\left(\frac{dN_2}{V}\right)^{1/2} = \left(\frac{\omega}{\pi^2 c^3}\right)^{1/2} d\omega \quad - (20)$$

From eq. (18):

$$\left(\frac{dN_3}{V}\right)^{1/3} = \frac{d\omega}{(3\pi^2 c^3)^{1/3}} \quad -(21)$$

Integrating eqs. (19) and (21) over frequency:

$$\int\left(\frac{dN_2}{V}\right)^{1/2} = \int\left(\frac{\omega}{\pi^2 c^3}\right)^{1/2} d\omega$$

$$= \frac{2\omega^{3/2}}{3(\pi^2 c^3)^{1/2}} \quad -(22)$$

so:

$$\frac{N_A}{V} = \left(\int\left(\frac{dN_2}{V}\right)^{1/2}\right)^2 = \frac{4\omega^3}{9\pi^2 c^3} \quad -(23).$$

Similarly: -(24)

$$\int\left(\frac{dN_3}{V}\right)^{1/3} = \int\frac{d\omega}{(3\pi^2 c^3)^{1/3}} = \frac{\omega}{(3\pi^2 c^3)^{1/3}}$$

so:

$$\frac{N_B}{V} = \left(\int\left(\frac{dN_3}{V}\right)^{1/3}\right)^3 = \frac{\omega^3}{3\pi^2 c^3} \quad -(25)$$

The total number of oscillators per volume of radiation is:

$$\frac{N}{V} = \frac{1}{V}\left(N_1 + N_A + N_B \right) \quad -(26)$$

$$= \left(\frac{10}{9} \frac{\omega^3}{\pi^2 c^3} \right)$$

The original Rayleigh result is eq. (4):

$$\frac{N}{V} = \frac{\omega^3}{3\pi^2 c^3} \quad -(27)$$

Conclusion

The original Rayleigh result is too small by a factor of $3/10 = 0.3$.

The correct result (26) is obtained from the correct infinitesimal density of oscillator states:

$$\boxed{\frac{dN}{V} = \frac{10}{3} \frac{\omega^2}{\pi^2 c^3} d\omega} \quad -(28)$$

So:

$$\frac{dN}{V} = \frac{\omega^2 d\omega}{2c^3\pi^2} + \frac{\omega (d\omega)^2}{2c^3\pi^2} + \frac{(d\omega)^3}{6c^3\pi^2} = \frac{10}{3}\frac{\omega^2}{\pi^2 c^3} d\omega \quad -(29)$$