279(4): Calculation of the Intensities of Reflected and Refracted Light

The energy of the photon as postulated by Planck is:

$$E_n = \hbar\omega, 2\hbar\omega, \ldots, n\hbar\omega \quad (n = 0, 1, 2 \ldots) \quad -(1)$$

The average energy of a beam of n photons all at the frequency ω is:

$$\langle E \rangle = \sum_n P_n E_n \quad -(2)$$

where the probability that the system has an energy E_n at temperature T is the Boltzmann distribution:

$$P_n = \frac{\exp(-E_n / kT)}{\sum_n \exp(-E_n / kT)} \quad -(3)$$

which leads to thermodynamic equilibrium. So

$$\langle E \rangle = \frac{\sum_n E_n \exp(-E_n / kT)}{\sum_n \exp(-E_n / kT)} \quad -(4)$$

$$\text{So } \langle E \rangle = \hbar\omega \frac{\sum_n n e^{-nx}}{\sum_n e^{-nx}} \quad -(5)$$

2)

where:

$$x = \exp\left(-\frac{\hbar\omega}{kT}\right) \quad - (6)$$

Eq. (5) may be written as:

$$\langle E \rangle = \hbar\omega \left(\frac{\sum_n n x^n}{\sum_n x^n} \right) \quad - (7)$$

If

$$\hbar\omega \ll kT \quad - (8)$$

then

$$\sum_n x^n = 1 + x + x^2 + \dots = (1-x)^{-1} \quad - (9)$$

and

$$\sum_n n x^n = x \frac{d}{dx} \left(\sum_n x^n \right) \quad - (10)$$

So

$$\langle E \rangle = \hbar\omega (1-x)\, x \frac{d}{dx} \left(\frac{1}{1-x} \right) \quad - (11)$$

$$\langle E \rangle = \left(\frac{x}{1-x} \right) \hbar\omega \quad - (12).$$

The intensity of the beam is:

$$\langle I \rangle = \frac{1}{c} \langle U \rangle \quad - (13)$$

([illegible] a beam with initial $\langle E \rangle$

where U is the energy density

3) The energy density is calculated from the Planck distribution and dN, the number density of oscillators in the range ω to $\omega + d\omega$. The energy density in this range is $\langle E \rangle dN$:

$$dU = \langle E \rangle dN \quad - (14)$$

the density of states method of Rayleigh and Jeans leads to:

$$dN = \frac{8\pi f^2}{c^3} df \quad - (15)$$

where

$$\omega = 2\pi f \quad - (16)$$

so the Planck distribution is:

$$dU = \langle E \rangle dN = \frac{8\pi h f^3}{c^3} \left(\frac{x}{1-x} \right) df \quad - (17)$$

where

$$\hbar = \frac{h}{2\pi}$$

So

$$dU = \langle E \rangle dN = \frac{\hbar}{\pi^2} \left(\frac{\omega}{c} \right)^3 \left(\frac{x}{1-x} \right) d\omega \quad - (18)$$

the energy density is therefore defined by:

$$U = \frac{\hbar}{\pi^2 c^3} \int_0 \omega^3 \left(\frac{x}{1-x} \right) d\omega \quad - (19)$$

4) where N is the number of frequencies in the beam. The intensity of the beam is therefore:

$$\boxed{I = cU = \frac{\hbar}{\pi^2 c^2} \int \omega^3 \left(\frac{x}{1-x} \right) d\omega} \quad -(20)$$

where

$$x = \frac{\hbar \omega}{kT} \quad -(21)$$

So the intensity of the refracted beam, I in watts per square metres, depends only on the frequency for a given temperature T.

If the refracting medium is an absorbing medium, then the Beer Lambert law means that:

$$I = I_0 \exp(-\alpha Z) \quad -(22)$$

where I_0 is the initial intensity and where I is the intensity after a distance Z, the sample thickness. The power absorption coefficient is

$$\alpha = \frac{\omega \epsilon''}{n' c} \quad -(23)$$

So it is clear that the frequency ω changes as the beam passes through the sample.

5) If $\hbar\omega \ll kT$ — (24)

then the average energy of a beam of many photons of different frequency is:

$$\langle E \rangle \sim \frac{\hbar}{\pi^2 c^3} \int \omega^3 \exp\left(-\frac{\hbar\omega}{kT}\right) d\omega$$

— (25)

It is again clear that the energy of the incoming, reflected and refracted beams cannot be the same.