279(5): Conservation of Energy and Momentum in a Refracted and Reflected Monochromatic Beam of n Photons.

Conservation of Total Energy

In the simplest theory, an incoming beam of n photons each of frequency ω is refracted to frequency ω_1 and reflected to frequency ω_2. The average energy of the incoming beam is:

$$\langle \hbar\omega \rangle = \left(\frac{x}{1-x} \right) \hbar\omega \quad -(1)$$

where

$$x = \exp\left(-\frac{\hbar\omega}{kT} \right) \quad -(2)$$

in the notation of previous notes.

If

$$\hbar\omega << kT \quad -(3)$$

then: as used in the derivation of eq. (1),

$$\langle \hbar\omega \rangle = \hbar\omega \exp\left(-\frac{\hbar\omega}{kT} \right) \quad -(4)$$

The average energy of the refracted beam is:

$$\langle \hbar\omega_1 \rangle = \left(\frac{x_1}{1-x_1} \right) \hbar\omega_1 \quad -(5)$$

where

$$x_1 = \frac{\hbar\omega_1}{kT} \quad -(6)$$

and the average energy of the reflected beam is:

$$\langle \hbar\omega_2 \rangle = \left(\frac{x_2}{1-x_2}\right) \hbar\omega_2 \quad -(7)$$

where

$$x_2 = \frac{\hbar\omega_2}{RT} \quad -(8)$$

It has been assumed that the temperature T of all three beams is the same.

So the equation of conservation of energy is:

$$\boxed{\langle \hbar\omega \rangle = \langle \hbar\omega_1 \rangle + \langle \hbar\omega_2 \rangle} \quad -(9)$$

where the mean energies are defined by eqs. (4), (5) and (7). So:

$$\left(\frac{x}{1-x}\right)\omega = \left(\frac{x_1}{1-x_1}\right)\omega_1 + \left(\frac{x_2}{1-x_2}\right)\omega_2 \quad -(10)$$

Conservation of Total Momentum

Define the incoming wave vector by:

$$\underline{\kappa} = \kappa_x \underline{i} + \kappa_y \underline{j} \quad -(11)$$

where

$$\kappa_x = \kappa \overset{\sin}{\cancel{\cos}}\,\theta \quad -(12)$$

$$\kappa_y = \kappa \overset{\cos}{\cancel{\sin}}\,\theta \quad -(13)$$

where θ is the angle of incidence. Then:

$$\underline{\kappa} = \kappa \left(\underline{i} \overset{\sin}{\cancel{\cos}}\,\theta + \underline{j} \overset{\cos}{\cancel{\sin}}\,\theta \right) \quad -(14)$$

3) If the incoming medium is air, then to an excellent approximation:

$$\kappa = \frac{\omega}{c} \qquad -(15)$$

Therefore the wave vector averaged over a Boltzmann distribution of n photons is:

$$\langle \underline{\kappa} \rangle = \frac{\langle \omega \rangle}{c}\left(\underline{i}\,\overset{\sin}{\cancel{\cos}}\,\theta + \underline{j}\,\overset{\cos}{\cancel{\sin}}\,\theta \right) \qquad -(16)$$

$$= \left(\frac{x}{1-x} \right) \frac{\omega}{c} \left(\underline{i}\,\overset{\sin}{\cancel{\cos}}\,\theta + \underline{j}\,\overset{\cos}{\cancel{\sin}}\,\theta \right) \qquad -(17)$$

Similarly:

$$\langle \underline{\kappa}_1 \rangle = \left(\frac{x_1}{1-x_1} \right) \frac{\omega_1}{v_1} \left(\underline{i}\,\overset{\sin}{\cancel{\cos}}\,\theta_1 + \underline{j}\,\overset{\cos}{\cancel{\sin}}\,\theta_1 \right) \qquad -(18)$$

where

$$v_1^2 = \frac{1}{\epsilon \mu}$$

is the phase velocity in the medium of refraction, with permittivity ϵ and permeability μ. The angle of refraction is θ_1.

Thirdly:

$$\langle \underline{\kappa}_2 \rangle = \left(\frac{x_2}{1-x_2} \right) \frac{\omega_2}{c} \left(\underline{i}\,\overset{\sin}{\cancel{\cos}}\,\theta_2 - \underline{j}\,\overset{\cos}{\cancel{\sin}}\,\theta_2 \right) \qquad -(19)$$

where θ_2 is the angle of reflection.

4) <u>By the</u> <u>experimental</u> <u>Snell</u> <u>Law</u>:

$$\theta = \theta_2 \quad -(20)$$

i.e. the angle of incidence is equal to the angle of reflection, and:

$$\sin\theta = n_1 \sin\theta_1 \quad -(21)$$

where

$$n_1^2 = \frac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0} \quad -(22)$$

is the refractive index in the medium in which refraction takes place, for example water.

The law of conservation of total linear momentum is:

$$\boxed{\hbar \langle \underline{\kappa} \rangle = \hbar \langle \underline{\kappa}_1 \rangle + \hbar \langle \underline{\kappa}_2 \rangle} \quad -(23)$$

From eq. (23):

$$\langle \underline{\kappa}_2 \rangle = \langle \underline{\kappa} \rangle - \langle \underline{\kappa}_1 \rangle \quad -(24)$$

so:

$$\begin{aligned} &\langle \underline{\kappa}_2 \rangle \cdot \langle \underline{\kappa}_2 \rangle \\ &= \langle \underline{\kappa} \rangle \cdot \langle \underline{\kappa} \rangle + \langle \underline{\kappa}_1 \rangle \cdot \langle \underline{\kappa}_1 \rangle \\ &\quad - 2 \langle \underline{\kappa} \rangle \cdot \langle \underline{\kappa}_1 \rangle \cos\theta_3 \end{aligned} \quad -(25)$$

where θ_3 is the angle between $\underline{\kappa}$ and $\underline{\kappa}_1$, i.e. the angle between the incoming and refracted beams.

From eqs. (16), (18) and (19):

$$\langle \underline{\kappa} \rangle \cdot \langle \underline{\kappa} \rangle = \left(\frac{x}{1-x} \right)^2 \frac{\omega^2}{c^2} \quad - (26)$$

$$\langle \underline{\kappa}_1 \rangle \cdot \langle \underline{\kappa}_1 \rangle = \left(\frac{x_1}{1-x_1} \right)^2 \frac{\omega_1^2}{V_1^2} \quad - (27)$$

$$\langle \underline{\kappa}_2 \rangle \cdot \langle \underline{\kappa}_2 \rangle = \left(\frac{x_2}{1-x_2} \right)^2 \frac{\omega_2^2}{c^2} \quad - (28)$$

and

$$\langle \underline{\kappa} \rangle \cdot \langle \underline{\kappa}_1 \rangle = \left(\frac{x}{1-x} \right) \left(\frac{x_1}{1-x_1} \right) \left(\frac{\omega}{c} \right) \left(\frac{\omega_1}{V_1} \right) \cdot \left(\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \right) \quad - (29)$$

Therefore the equation of conservation of momentum (23) gives:

$$\left(\frac{x_2}{1-x_2} \right)^2 \frac{\omega_2^2}{c^2} = \left(\frac{x}{1-x} \right)^2 \frac{\omega^2}{c^2} + \left(\frac{x_1}{1-x_1} \right)^2 \frac{\omega_1^2}{V_1^2} - 2 \left(\frac{x}{1-x} \right) \left(\frac{x_1}{1-x_1} \right) \left(\frac{\omega \omega_1}{c V_1} \right) \left(\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \right) \quad - (\overset{30}{\cancel{24}})$$

Eqs. (10) and (24) can be solved

simultaneously to express ω in terms of ω_1 or vice versa. This gives the refracted frequency in terms of the incident frequency. Eqs. (10) and ($\cancel{24}$ 30) can also be solved simultaneously to give the reflected frequency ω_2 in terms of the incident frequency ω.

In these equations:

$$v_1^2 = \frac{1}{\epsilon_1\mu_1} \quad - (\cancel{25}\ 31)$$

and

$$c^2 = \frac{1}{\epsilon_0\mu_0} \quad - (\cancel{26}\ 32)$$

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so

$$\frac{1}{v_1^2} = \epsilon_1\mu_1 \quad - (\cancel{27})$$

and

$$\frac{1}{c v_1} = (\mu\epsilon\mu_0\epsilon_0)^{1/2} \quad - (\cancel{28}\ 33)$$

Fm eqs ($\cancel{25}$ 31) and ($\cancel{26}$ 32):

$$n_1^2 = \frac{\epsilon_1\mu_1}{\epsilon_0\epsilon_0} = \left(\frac{c}{v_1}\right)^2 \quad - (\cancel{29}\ 34)$$

so

$$v_1 = \frac{c}{n_1} \quad - (\cancel{30}\ 35)$$

and

$$CV_1 = \frac{c^2}{n_1} \quad -(36)$$

So

$$\frac{1}{CV_1} = \frac{n_1}{c^2} \quad -(37)$$

and

$$\frac{1}{V_1^2} = \frac{n_1^2}{c^2} \quad -(38)$$

Therefore eq. (24) is:

$$\left(\frac{x_2}{1-x_2}\right)^2 \omega_2^2 = \left(\frac{x}{1-x}\right)^2 \omega^2 + n_1^2 \left(\frac{x_1}{1-x_1}\right)^2 \omega_1^2 - 2\left(\frac{x}{1-x}\right)\left(\frac{x_1}{1-x_1}\right) n_1 \omega \omega_1 \left(\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1\right) \quad -(39)$$

In general the refractive index is complex valued:

$$n_1 = n_1' + i n_1'' \quad -(\cancel{35}\ 40)$$

so:

$$n_1'^2 = \frac{1}{2}\left(\epsilon_{1r}' + \left(\epsilon_{1r}'^2 + \epsilon_{1r}''^2\right)^{1/2}\right) \quad -(\cancel{36}\ 41)$$

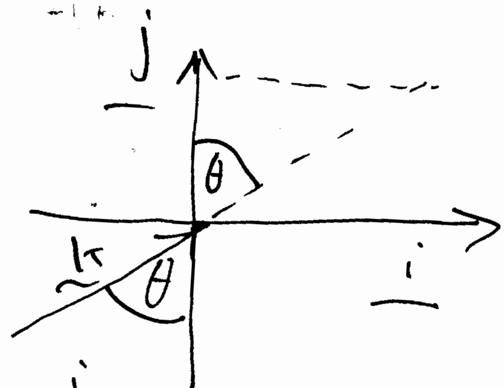
$$n_1'' = \frac{\epsilon_{1r}''}{2 n_1'} \quad -(\cancel{37}\ 42)$$

and

$$n_1'^2 - n_1''^2 = \epsilon_{1r}' \quad -(\cancel{38}\ 43)$$

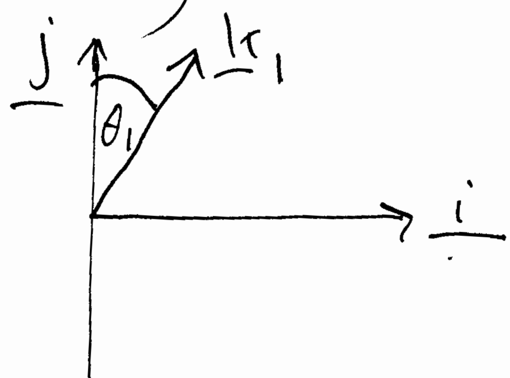
8) <u>Vector Diagrams</u>

1) <u>Incoming</u>



$$\underline{k} = k_x\,\underline{i} + k_y\,\underline{j}$$
$$= k\left(\underline{i}\sin\theta + \underline{j}\cos\theta\right)$$

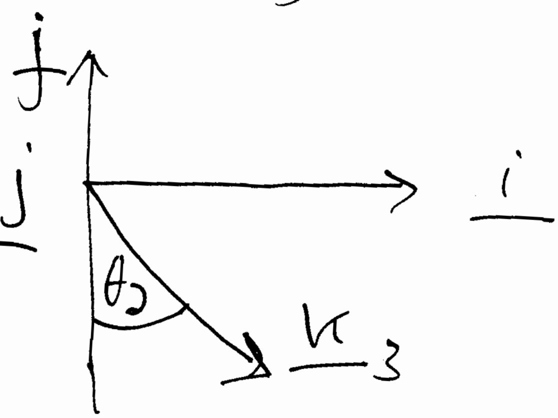
2) <u>Refracted</u>



$$\underline{k}_1 = k_{x1}\,\underline{i} + k_{y1}\,\underline{j}$$
$$= k_1\left(\underline{i}\sin\theta_1 + \underline{j}\cos\theta_1\right)$$

3) <u>Reflected</u>

$$\underline{k}_2 = k_2\sin\theta_2\,\underline{i} - k_2\cos\theta_2\,\underline{j}$$
$$= k_{x2}\,\underline{i} - k_{y2}\,\underline{j}$$



$$\underline{k}\cdot\underline{k}_1 = k k_1\left(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\right)$$
$$= k k_1\cos\left(\theta - \theta_1\right) = k k_1\cos\theta_3$$