276(5): Development of Thomas Precession Theory

The three dimensional metric can be written as:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - r^2 d\beta^2 - dr^2 \quad -(1)$$

where

$$d\beta^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \quad -(2)$$

The velocity is defined by:

$$v^2 = r^2 \left(\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\phi}{dt}\right)^2 \right) \quad -(3)$$

$$= r^2 \left(\frac{d\beta}{dt}\right)^2$$

The two dimensional metric is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(4),$$

and the two dimensional Thomas precession is defined by:

$$\phi' = \phi + \omega t \quad -(5)$$

so the three dimensional analogy is:

$$\beta' = \beta + \omega t \quad -(6)$$

As shown in UFT 265 eq. (5) results in:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{3 v_\theta^2}{c^2}\right) c^2 dt^2 - v^2 dt^2 \quad -(7)$$

$$= \left(1 - \frac{6MG}{c^2 r}\right) c^2 dt^2 - v^2 dt^2$$

Therefore:

$$c^2 d\tau^2 \left(1 - \frac{6MG}{c^2 r}\right)^{-1} = c^2 dt^2 - v^2 \left(1 - \frac{6MG}{c^2 r}\right)^{-1} \quad -(8)$$

Therefore the proper time is increased by a factor of:

$$\tau' = \tau \left(1 - \frac{6MG}{c^2 r}\right)^{-1/2} \quad -(9)$$

$$\sim \tau \left(1 + \frac{3MG}{c^2 r}\right)$$

and the total velocity v is increased by:

$$v' = v \left(1 - \frac{6MG}{c^2 r}\right)^{-1/2} \quad -(10)$$

$$\sim v \left(1 + \frac{3MG}{c^2 r}\right)$$

The velocity v is also the classical velocity, so denoting:

$$x = 1 + \frac{3MG}{c^2 r} \quad -(11)$$

the hamiltonian is changed to:

$$H' = E' = \frac{1}{2} m x^2 \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2\right) - \frac{k}{r} \quad -(12)$$

compared with the original:

$$H = E = \frac{1}{2} m \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2\right) - \frac{k}{r} \quad -(13)$$

5) The lagrangian is changed to:

$$\mathcal{L}' = \frac{1}{2} m x^2 \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 \right) + \frac{k}{r} \quad -(14)$$

compared with the original:

$$\mathcal{L} = \frac{1}{2} m \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 \right) + \frac{k}{r}. \quad -(15)$$

From the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(16)$$

The angular velocity from eq. (15) is:

$$\frac{d\phi}{dt} = \frac{L}{mr^2}, \quad -(17)$$

and using

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} \quad -(18)$$

eq. (13) becomes:

$$E = \frac{1}{2}\frac{L^2}{mr^4}\left(\left(\frac{dr}{d\phi}\right)^2 + r^2 \right) - \frac{k}{r} \quad -(19)$$

i.e.

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{2mE}{L^2}\right) r^4 - r^2 + \left(\frac{2mk}{L^2}\right) r^3 \quad -(20)$$

Eq. (20) is also given by the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\phi}. \quad -(21)$$

4) This follows from:

$$\frac{dr}{d\phi} = \frac{\epsilon}{\alpha} r^2 \sin\phi \quad -(22)$$

so

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\epsilon}{\alpha} r^2 \left(1 - \cos^2\phi\right) \quad -(23)$$

where

$$\cos^2\phi = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(24)$$

It follows ~~that~~:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{\alpha^2}\left(\epsilon^2 - 1\right) - r^2 + \frac{2r^3}{\alpha} \quad -(25)$$

which is eq. (20) provided ~~that~~:

$$\frac{\epsilon^2 - 1}{\alpha^2} = \frac{2mE}{L^2} \quad -(26)$$

and

$$\alpha = \frac{L^2}{mk} \quad -(27)$$

When v is replaced by xv, eq. (16) gives:

$$\frac{d}{dt}\left(x^2 m r^2 \frac{d\phi}{dt}\right) = 0 \quad -(28)$$

so the conserved angular momentum is changed to:

$$L \rightarrow x^2 L = L' \quad -(29)$$

Therefore:

5)

$$\frac{d\phi}{dt} = \frac{L}{mr^2 x^2} \quad -(30)$$

and

$$E = \frac{1}{2} m x^2 \left(\frac{d\phi}{dt}\right)^2 \left(\left(\frac{dr}{d\phi}\right)^2 + r^2\right) - \frac{k}{r}$$

$$= \frac{1}{2} \frac{L^2}{mr^4 x^2} \left(\left(\frac{dr}{d\phi}\right)^2 + r^2\right) - \frac{k}{r} \quad -(31)$$

so:

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{2mE}{x^2 L^2}\right) r^4 - r^2 + \left(\frac{2mk}{x^2 L^2}\right) r^3 \quad -(32)$$

Now consider the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\phi)} \quad -(33)$$

It follows that:

$$\frac{dr}{d\phi} = \left(\frac{\epsilon x}{\alpha}\right) r^2 \sin(x\phi) \quad -(34)$$

and

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\epsilon^2 x^2}{\alpha^2} r^4 \left(1 - \cos^2(x\phi)\right) \quad -(35)$$

where

$$\cos(x\phi) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(36)$$

so:

$$\left(\frac{dr}{d\phi}\right)^2 = x^2 \frac{(\epsilon^2 - 1)}{\alpha^2} r^4 - x^2 r^2 + \frac{2x^2}{\alpha} r^3 \quad -(37)$$

6)

Eqs. (32) and (37) are the same if:

$$\frac{x^2(\epsilon^2-1)}{\alpha^2} = \frac{2mE}{x^2L^2}, \quad -(38)$$

$$\frac{mk}{x^2L^2} = \frac{x^2}{\alpha} \quad -(39)$$

and

$$xr \sim r \quad -(40).$$

Under conditions (38) to (40) the 2D Thomas precession gives the precessing elliptical orbit, QED.

By experimental observation:

$$x = 1 + \frac{3MG}{c^2\alpha} \quad -(41)$$

so

$$x = \left(1 + \frac{3MG}{c^2 r}\right)_{r=\alpha} \quad -(42)$$

The condition

$$r = \alpha \quad -(43)$$

means that

$$m\frac{d^2r}{dt^2} = -\frac{dV(r)}{dr} = 0 \quad -(44)$$

where the effective potential is:

$$V(r) = -\frac{mMG}{r} + \frac{L^2}{2mr^2} \quad -(45)$$

7) Under condition (45) and (44) the mass m behaves as if it is a free mass or free particle, and this is consistent with the fact that the starting metric is ~~that~~ of a free particle.

So the 2D Thomas precession gives the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\phi)} \quad -(46)$$

where

$$x = 1 + \frac{3mG}{c^2 \alpha} \quad -(47)$$

This theory must now be developed in three dimensions.
