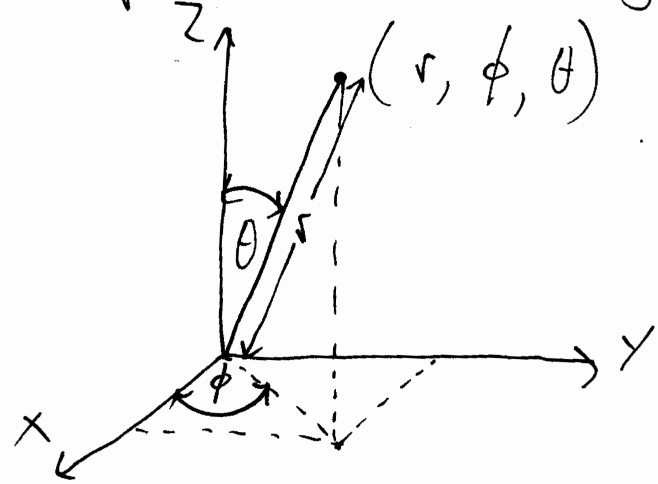
276(4): Three Dimensional Thomas Precession.

Consider the spherical polar coordinate system:

$$X = r\sin\theta\cos\phi \quad -(1)$$
$$Y = r\sin\theta\sin\phi \quad -(2)$$
$$Z = r\cos\theta \quad -(3)$$




It follows that:

$$dX = -r\sin\theta\cos\phi\, d\phi + r\cos\theta\cos\phi\, d\theta + \sin\theta\cos\phi\, dr.$$
$$dY = r\sin\theta\cos\phi\, d\phi + r\cos\theta\sin\phi\, d\theta + \sin\theta\sin\phi\, dr$$
$$dZ = -r\sin\theta\, d\theta + \cos\theta\, dr \quad -(4)$$

So:

$$ds^2 = dX^2 + dY^2 + dZ^2$$
$$= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad -(5)$$

The infinitesimal line element of special relativity is therefore:

$$ds^2 = c^2 d\tau^2 \underset{\,}{\overset{= c^2 dt^2}{/}} - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad -(6)$$

This can be written as:

$$ds^2 = (c^2 - v^2)\, dt^2 = c^2 d\tau^2 \quad -(7)$$

where:

$$v^2 dt^2 = r^2 (d\theta^2 + \sin^2\theta\, d\phi^2), \quad -(8)$$

so

$$v^2 = r^2 \left( \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\phi}{dt}\right)^2 \right) \quad -(9)$$

as in previous notes.

In UFT 265 the Thomas precession was shown to be the cause of planet precession for planar orbits. In that theory the 3D infinitesimal line element (6) reduces to:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - r^2 d\phi^2 - dr^2 \quad -(10)$$

The Thomas precession is generated by:

$$\phi' = \phi + \omega t \quad -(11)$$

in 2D theory. As in UFT 265 this procedure results in:

$$\frac{d\tau}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} = \left(1 - 3\frac{v_\phi^2}{c^2} - \frac{v^2}{c^2}\right)^{1/2} \quad -(12)$$

) where:

$$\underline{v}_\phi = \underline{\omega} \times \underline{r}. \quad -(13)$$

In the classical limit:

$$v \ll c, \quad v_\phi \ll c \quad -(14)$$

the Lorentz factor becomes:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{3v_\phi^2}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \quad -(15)$$

$$\sim 1 + \frac{v^2}{2c^2} + \frac{3v_\phi^2}{2c^2}$$

The total velocity is:

$$\underline{v} = \underline{v}_r + \underline{\omega} \times \underline{r} \quad -(16).$$

where

$$\underline{v}_r = \frac{dr}{dt}\underline{e}_r. \quad -(17)$$

and

$$\underline{v}_\phi = \underline{\omega} \times \underline{r} \quad -(18)$$

As in UFT265 the effect of eq. (11) is to produce:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{3v_\phi^2}{c^2}\right)c^2 dt^2 - v^2 dt^2 \quad -(19)$$

$$= \left(1 - \frac{3v_\phi^2}{c^2}\right)c^2 dt^2 - dX^2 - dY^2 - dZ^2$$

Under the frame rotation:

4)

$$dx^2 + dy^2 + dz^2 = \text{constant} \quad -(20)$$

so the overall effect is:

$$dt'^2 = \left(1 - \frac{3v_\phi^2}{c^2}\right) dt^2$$

$$= \left(1 - \frac{6mG}{rc^2}\right) dt^2 \quad -(21)$$

If:

$$6mG << rc^2 \quad -(22)$$

then

$$dt' = \left(1 - \frac{6mG}{rc^2}\right)^{1/2} dt \quad -(23).$$

i.e.

$$dt \sim \left(1 - \frac{6mG}{rc^2}\right)^{-1/2} dt' \quad -(24)$$

$$dt \sim \left(1 + \frac{3mG}{rc^2}\right) dt' \quad -(25)$$

Therefore:

$$\boxed{\frac{dt}{dt'} \sim 1 + \frac{3mG}{rc^2}} \quad -(26)$$

The angular velocity of the frame rotation is:

$$\omega = \frac{d\phi}{dt} \quad -(27)$$

) and this can be written as:

$$\omega = \frac{d\phi}{\left(1 + \frac{3mG}{rc^2}\right) dt'} \quad - (28)$$

So if $$d\phi = \left(1 + \frac{3mG}{rc^2}\right) d\phi' \quad - (29)$$

the angular velocity is constant:

$$\omega = \frac{d\phi'}{dt'} = \frac{d\phi}{dt} \quad - (30)$$

Therefore: $$\boxed{\frac{d\phi}{d\phi'} = 1 + \frac{3mG}{rc^2}} \quad - (31)$$

The observed precession of orbits is:

$$x = 1 + \frac{3mG}{\alpha c^2} \quad - (32)$$

so eqs. (31) and (32) are the same if:

$$r = \alpha \quad - (33)$$

So at the turning point:

$$\boxed{\frac{L}{L_Z} = \frac{d\phi}{d\phi'} = 1 + \frac{3mG}{\alpha c^2}} \quad - (34)$$

) where $L/L_Z$ comes from classical three dimensional orbit theory. So $L/L_Z$ is the classical limit of the Thomas precession theory. For the two dimensional ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(35)$$

the turning point

$$r = \alpha \quad -(36)$$

occurs when

$$m\frac{d^2 r}{dt^2} = -\frac{mMG}{r} + \frac{L_Z^2}{2mr^2} = 0 \quad -(37)$$

where

$$\alpha = \frac{L_Z^2}{m^2 MG} \quad -(38)$$

Eq. (37) is equivalent to:

$$\epsilon = 0 \quad -(39)$$

or

$$\cos\phi = 0. \quad -(40)$$

In a precessing ellipse the turning point is also eq. (36). For a precessing planar ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\phi)} \quad -(41)$$

$$m\frac{d^2r}{dt^2} = x^2\left(\frac{L_Z^2}{mr^3} - \frac{L_Z^2}{2mr^2}\right) \quad -(42)$$

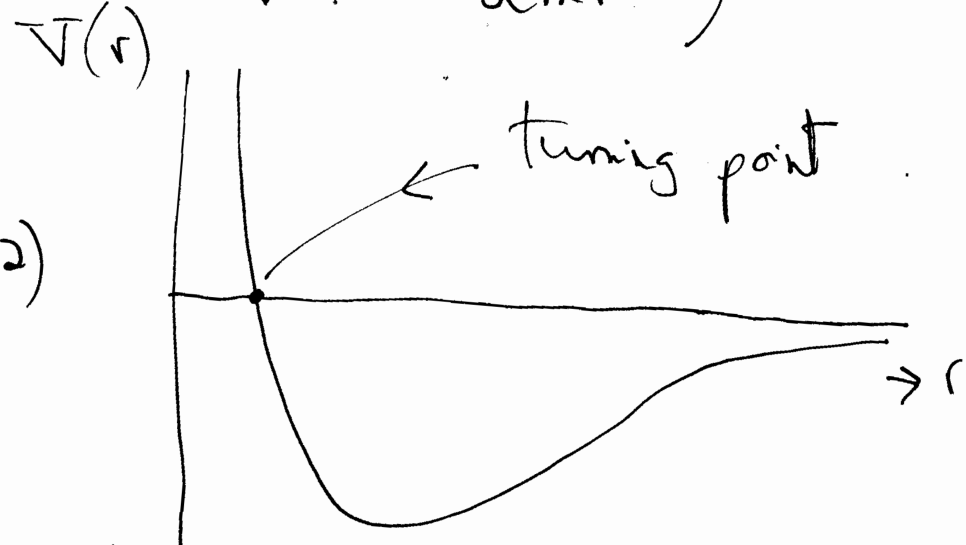



Fig (2)

The turning point is defined by:

$$V(r) = -\frac{mMG}{r} + \frac{L_Z^2}{2mr^2}$$

$$= 0 \quad -(43)$$

and is defined in Fig (2). The turning point has a universal significance because it is the point in the elliptic or precessing elliptic orbits where the net force is zero:

$$F = m\frac{d^2r}{dt^2} = 0 \quad -(44)$$

and it defines the precession constant through:

$$x = \left(1 + \frac{3MG}{rc^2}\right)_{r=\alpha} \quad -(45)$$

$$= \frac{L}{L_Z}$$