275(6): Summary of the General Theory of 3D Orbits

Consider the hamiltonian:

$$H = \frac{1}{2} m v^2 + U(r) \quad -(1)$$

and the lagrangian:

$$\mathcal{L} = \frac{1}{2} m v^2 - U(r) \quad -(2)$$

where U is any function of r. The solution of eq. (1) is:

$$\frac{1}{r} = f(\beta) \quad -(3)$$

where f is any function of β.

The force law equivalent to Eq. (1) is:

$$F(r) = -\frac{L^2}{m r^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(4)$$

where:

$$m\ddot{r} = -\frac{L^2}{m r^2}\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) \quad -(5)$$

The transition from 2D to 3D orbital theory takes place through a transition in the kinetic energy:

$$T = \frac{1}{2} m v^2, \quad -(6)$$

the potential energy $U(r)$ is the same in 2D and 3D.

In 2D:

$$v^2 = \dot{r}^2 + \dot{\phi}^2 r^2 \quad -(7)$$

and in 3D:

2)

$$v^2 = \dot{r}^2 + \dot{\beta}^2 r^2 \quad -(8)$$

where:

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(9)$$

Eqs. (7) to (9) lead to:

$$\tan\phi = \frac{L_Z}{L}\tan\beta \quad -(10)$$

and

$$\cos\theta = \left(1 - \left(\frac{L_Z}{L}\right)^2\right)^{1/2}\sin\beta \quad -(11)$$

i.e:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L}{L_Z}\right)^2\sin^2\phi\right)^{1/2}}, \quad -(12)$$

$$\cos\beta = \left(1 - \left(\frac{1}{1-\left(\frac{L_Z}{L}\right)^2}\right)\cos^2\theta\right)^{1/2} \quad -(13)$$

Conclusion

In general, a three dimensional orbit is given by Eq. (3) with β defined by eq. (12). Therefore r may be expressed in terms of ϕ, in terms of θ, or a combination of ϕ and θ by adding eqs. (12) and (13):

$$\cos\beta = \frac{1}{2}\left[\frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L}{L_z}\right)^2 \sin^2\phi\right)^{1/2}} + \left(1 - \left(\frac{1}{1-\left(\frac{L_z}{L}\right)^2}\right)\cos^2\theta\right)^{1/2}\right]$$

Eq. (14) gives an orbit $r(\theta, \phi)$ — (14) for any β and any force law or potential $u(r)$!

Example

$$U(r) = -\frac{L^2}{2m}\left(1+\alpha^2\right)\frac{1}{r^2}. \quad -(15)$$

This gives:

$$F(r) = -\frac{L^2}{mr^3}\left(1+\alpha\right)^2 \quad -(16)$$

and the logarithmic spiral orbit:

$$r = r_0 \exp(\alpha\beta) \quad -(17)$$

using eq. (4)

Using

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(18)$$

4) it is found that:

$$\beta(t) = \frac{1}{2\alpha} \log_e \left(\frac{2\alpha L t}{m r_0^2} \right) \quad -(19)$$

and
$$r(t) = \left(\frac{2\alpha L t}{m} \right)^{1/2} . \quad -(20)$$

therefore from eq^s. (10) and (11):

$$\phi(t) = \tan^{-1} \left(\frac{L_Z}{L} \tan \beta(t) \right) \quad -(21)$$

and
$$\theta(t) = \cos^{-1} \left(\left(1 - \left(\frac{L_Z}{L} \right)^2 \right)^{1/2} \sin \beta(t) . \right) \quad -(22).$$
