275(5): Three Dimensional Hyperbolic Spiral Galaxy

In this case:
$$r = \frac{r_0}{\beta} \quad -(1)$$

and
$$\frac{d\beta}{dt} = \frac{L}{mr^2} = \frac{L}{mr_0^2}\beta^2 \quad -(2)$$

So:
$$\frac{dt}{d\beta} = \left(\frac{mr_0^2}{L}\right)\frac{1}{\beta^2} \quad -(3)$$

and
$$t = \frac{mr_0^2}{L}\int\frac{d\beta}{\beta^2} = -\frac{mr_0^2}{L\beta} \quad -(4)$$

Therefore:
$$\beta(t) = -\frac{mr_0^2}{Lt} \quad -(5)$$

and
$$r(t) = -\frac{L}{mr_0}t \quad -(6)$$

Now use the kinetic energy relations:
$$\tan\phi = \frac{L_Z}{L}\tan\beta \quad -(7)$$

and
$$\cos\theta = \left(1-\left(\frac{L_Z}{L}\right)^2\right)^{1/2}\sin\beta \quad -(8)$$

to find:
$$\phi(t) = \tan^{-1}\left(\frac{L_Z}{L}\tan\beta(t)\right) \quad -(9)$$

and

2)

$$\theta(t) = \cos^{-1}\left(\left(1-\left(\frac{L_z}{L}\right)^2\right)^{1/2}\sin\beta(t)\right) \quad -(10)$$

where:

$$\beta(t) = -\frac{mr_0^2}{Lt} \quad -(11)$$

It is also possible to express r in terms of ϕ using eqns. (1) and (7):

$$r = \frac{r_0}{\tan^{-1}\left(\frac{L}{L_z}\tan\phi\right)} \quad -(12)$$

and r in terms of θ using eqs. (1) and (8):

$$r = \frac{r_0}{\sin^{-1}\left(\left(1-\left(\frac{L_z}{L}\right)^2\right)^{-1/2}\cos\theta\right)} \quad -(13)$$

From eqns. (12) and (13) it is clear that r is a function of ϕ and θ as follows:

$$r = \frac{1}{2}r_0\left[\frac{1}{\tan^{-1}\left(\frac{L}{L_z}\tan\phi\right)} + \frac{1}{\sin^{-1}\left(\left(1-\left(\frac{L_z}{L}\right)^2\right)^{-1/2}\cos\theta\right)}\right] \quad -(14)$$