275(1): Simplest Formulation of the 3D Orbit for the Inverse Square Law.

The two basic equations of the orbit are:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(1)$$

and

$$z^2 = \left(1 - \frac{L_Z}{L}\right) y^2 \quad -(2)$$

where:

$$x = a\epsilon + r\cos\beta \quad -(3)$$

$$y = r\sin\beta \quad -(4)$$

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(5)$$

$$\alpha = a(1-\epsilon^2) \quad -(6)$$

$$\epsilon^2 = 1 - \frac{b^2}{a^2} \quad -(7)$$

$$H = \frac{1}{2}m(\dot{r}^2 + \dot{\beta}^2 r^2) - \frac{k}{r} \quad -(8)$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + \dot{\beta}^2 r^2) + \frac{k}{r} \quad -(9)$$

$$k = mMG \quad -(10)$$

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(11)$$

The angular equations are:

2)

$$\cos^2\theta = \left(\frac{L^2 - L_z^2}{L^2}\right)\sin^2\beta \quad -(12)$$

$$= \left(1 - \left(\frac{L_z}{L}\right)^2\right)\sin^2\beta$$

and

$$\tan\phi = \left(\frac{L_z}{L}\right)\tan\beta \quad -(13)$$

i.e.

$$\cos^2\beta = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L}{L_z}\right)^2\sin^2\phi} \quad -(14)$$

The angular relations (12) to (14) depend only on eq. (11), and are true for any potential energy U of the hamiltonian:

$$H = T + U \quad -(15)$$

and lagrangian:

$$\mathcal{L} = T - U. \quad -(16)$$

Divide eq. (2) by a distance c yet to be determined:

$$\frac{Z^2}{c^2} = \left(1 - \frac{L_z}{L}\right)\frac{Y^2}{c^2} \quad -(17)$$

3)

Add equations (1) and (17):

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1 + \left(1 - \frac{L_Z}{L}\right)\frac{Y^2}{c^2} \quad -(18)$$

This is the ellipsoidal orbit:

$$\frac{X^2}{a^2} + Y^2\left(\frac{1}{b^2} - \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right)\right) + \frac{Z^2}{c^2} = 1 \quad -(19)$$

observed in three dimensional galaxies.

Reduction to Spherical Orbit.

The spherical orbit is:

$$X^2 + Y^2 + Z^2 = r^2 \quad -(20)$$

and eq. (19) becomes eq. (20) when:

$$\frac{1}{a^2} = \frac{1}{c^2} = \frac{1}{b^2} - \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right) = \frac{1}{r^2} \quad -(21)$$

i.e. $a = c = r \quad -(22)$

$$\frac{1}{b^2} - \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right) = \frac{1}{r^2} \quad -(23)$$

4) From eq. (23):

$$c = r \quad -(24)$$

so:

$$\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{L_Z}{L}\right) = \frac{1}{r^2} \quad -(25)$$

and

$$\frac{1}{b^2} = \frac{1}{r^2}\left(1 + 1 - \frac{L_Z}{L}\right)$$

$$= \frac{1}{r^2}\left(2 - \frac{L_Z}{L}\right) \quad -(26)$$

Therefore the spherical orbit (20) is obtained when:

$$a = c = r \quad -(27)$$

and

$$b = r\left(2 - \frac{L_Z}{L}\right)^{-1/2} \quad -(28)$$

<u>Elongated Ellipsoidal Orbit</u>

In this case:

$$a^2 \gg b^2 \sim c^2 \quad -(29)$$

The spherical and elongated ellipsoidal orbits are observed experimentally in three dimensional galaxies.

5) The semi major axis a and semi minor axis b of the Seta ellipse are defined by:

$$\alpha = \frac{L^2}{mk} \quad -(30)$$

and

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(31)$$

where L is the total angular momentum and E the total energy, two constants of motion. So:

$$\alpha = \frac{L^2}{mk} = a(1-\epsilon^2) \quad -(32)$$

and

$$\epsilon^2 = 1 - \frac{b^2}{a^2} = 1 + \frac{2EL^2}{mk^2} \quad -(33)$$

It follows that:

$$E = -\frac{b^2}{a^2}\frac{mk^2}{2L^2} \quad -(34)$$

and

$$\alpha = a(1-\epsilon^2) = \frac{b^2}{a} \quad -(35)$$

So:

$$L^2 = mk\alpha = \frac{mkb^2}{a} \quad -(36)$$

From eqs. (34) and (36):

$$E = -\frac{k}{2a} \quad -(37)$$

Therefore for the beta ellipse:

$$0 < \epsilon < 1 \quad -(38)$$

and

$$V_{min} < E < 0 \quad -(39)$$

where

$$V = -\frac{k}{r} + \frac{L^2}{2mr^2} \quad -(40)$$

Therefore:

$$|E| = \frac{k}{2a} \quad -(41)$$

and

$$\boxed{a = \frac{k}{2|E|} = \frac{mMG}{2|E|}} \quad -(42)$$

It follows that:

$$1 - \epsilon^2 = 2|E|\frac{\alpha}{k} \quad -(43)$$

and that:

$$\boxed{b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2m|E|)^{1/2}}}$$

$$-(44)$$