275(3): The Sixteen Fundamental 3D Orbits from the Inverse Square Law.

Ellipse

1)
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1 + \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right)Y^2 \quad -(1)$$

i.e.
$$\frac{X^2}{a^2} + Y^2\left(\frac{1}{b^2} - \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right)\right) + \frac{Z^2}{c^2} = 1 \quad -(2)$$

This is an ellipsoidal orbit.

2)
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - \frac{Z^2}{c^2} = 1 - \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right)Y^2 \quad -(3)$$

i.e.
$$\frac{X^2}{a^2} + Y^2\left(\frac{1}{b^2} + \frac{1}{c^2}\left(1 - \frac{L_Z}{L}\right)\right) - \frac{Z^2}{c^2} = 1 \quad -(4)$$

This is a one sheet hyperboloidal orbit

3)
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z}{c} = 1 + \frac{Y}{c}\left(1 - \frac{L_Z}{L}\right) \quad -(5)$$

4)
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - \frac{Z}{c} = 1 - \frac{Y}{c}\left(1 - \frac{L_Z}{L}\right) \quad -(6)$$

These are types of elliptic paraboloid orbits.

Hyperbola

5)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 + \left(1 - \frac{L_z}{L}\right)\frac{y^2}{c^2} \quad -(7)$$

i.e.
$$\frac{x^2}{a^2} - y^2\left(\frac{1}{b^2} + \frac{1}{c^2}\left(1 - \frac{L_z}{L}\right)\right) + \frac{z^2}{c^2} = 1 \quad -(8)$$

This is a second type of one sheet hyperboloidal orbit.

6)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \left(1 - \frac{L_z}{L}\right)\frac{y^2}{c^2} \quad -(9)$$

i.e.
$$\frac{x^2}{a^2} - y^2\left(\frac{1}{b^2} - \frac{1}{c^2}\left(1 - \frac{L_z}{L}\right)\right) - \frac{z^2}{c^2} = 1 \quad -(10)$$

This is a two sheet hyperboloidal orbit.

7)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z}{c} = 1 - \left(1 - \frac{L_z}{L}\right)\frac{y}{c} \quad -(11)$$

8)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z}{c} = 1 + \left(1 - \frac{L_z}{L}\right)\frac{y}{c} \quad -(12)$$

These are two types of hyperbolic paraboloid orbits.

) Parabola

$$\frac{Y^2}{a^2} + \frac{Z^2}{b^2} = \frac{4X}{a} + \frac{1}{b^2}\left(1-\frac{L_z}{L}\right)Y^2 \quad -(13)$$

i.e.

9) $$Y^2\left(\frac{1}{a^2} - \frac{1}{b^2}\left(1-\frac{L_z}{L}\right)\right) + \frac{Z^2}{b^2} = \frac{4X}{a} \quad -(14)$$

This is another type of elliptic parabaloid orbit.

$$\frac{Y^2}{a^2} - \frac{Z^2}{b^2} = \frac{4X}{a} - \frac{1}{b^2}\left(1-\frac{L_z}{L}\right)Y^2 \quad -(15)$$

i.e.

10) $$Y^2\left(\frac{1}{a^2} + \frac{1}{b^2}\left(1-\frac{L_z}{L}\right)\right) - \frac{Z^2}{b^2} = \frac{4X}{a} \quad -(16)$$

This is a hyperbolic parabaloid orbit.

11) $$\frac{Y^2}{a^2} + \frac{Z}{b} = \frac{4X}{a} - \frac{1}{b}\left(1-\frac{L_z}{L}\right)Y \quad -(17)$$

12) $$\frac{Y^2}{a^2} - \frac{Z}{b} = \frac{4X}{a} + \frac{1}{b}\left(1-\frac{L_z}{L}\right)Y \quad -(18)$$

These are types of parabaloid orbits

4)

Circle

13) $$x^2 + y^2\left(1 - \left(1 - \frac{L_z}{L}\right)^2\right) + z^2 = r^2 \quad -(19)$$

This is another type of ellipsoidal orbit.

14) $$x^2 + y^2\left(1 + \left(1 - \frac{L_z}{L}\right)^2\right) - z^2 = r^2 \quad -(20)$$

This is another type of one sheet hyperboloidal orbit.

15) $$x^2 + y^2 + aZ = r^2 - aY\left(1 - \frac{L_z}{L}\right) \quad -(21)$$

This is another type of ellipsoidal parabolic orbit.

16) $$x^2 + y^2 - aZ = r^2 + aY\left(1 - \frac{L_z}{L}\right) \quad -(22)$$

This is another type of ellipsoidal parabolic orbit.

All these orbits are fundamental types of three dimensional conic sections given by:

$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(23)$$

where:

5)

$$\cos^2\beta = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L}{L_Z}\right)^2 \sin^2\phi} \quad -(24)$$

$$\sin^2\beta = \frac{1}{\left(1 - \left(\frac{L_Z}{L}\right)^2\right)} \cos^2\theta \quad -(25)$$

of the (r, θ, ϕ) spherical polar coordinate system.

Therefore:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L}{L_Z}\right)^2 \sin^2\phi\right)^{1/2}} \quad -(26)$$

and

$$\cos\beta = \left(1 - \left(\frac{1}{1 - \left(\frac{L_Z}{L}\right)^2}\right) \cos^2\theta\right)^{1/2} \quad -(27)$$

Adding eqs. (26) and (27):

$$\cos\beta = \frac{1}{2}\left[\frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L}{L_Z}\right)^2 \sin^2\phi\right)^{1/2}} + \left(1 - \left(\frac{1}{1 - \left(\frac{L_Z}{L}\right)^2}\right)\cos^2\theta\right)^{1/2}\right] \quad -(28)$$

Therefore r can be graphed as a

) function of ϕ and θ using eqns. (23) and (28).

Also r can be graphed as a function of ϕ and as a function of θ.

All sixteen fundamental orbits in Cartesian representation can be graphed as $r(\theta, \phi)$ using eqns. (23) and (28).

They are defined as follows from the beta conic section (23).

1) Beta Ellipse

$$0 < \epsilon < 1 \quad -(29)$$

This gives types (1) to (4) 3D orbits.

2) Beta Hyperbola

$$\epsilon > 1 \quad -(30)$$

This gives types (5) to (8) 3D orbits.

3) Beta Parabola

$$\epsilon = 1 \quad -(31)$$

This gives types (9) to (12) 3D orbits.

4) Beta Circle

$$\epsilon = 0 \quad -(32)$$

This gives types (13) to (16) 3D orbits.