275(4): Time Dependence and Equations for Animation

Consider the beta conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(1)$$

From the Lagrangian:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(2)$$

where L is the total angular momentum. So

$$\frac{d\beta}{dt} = \frac{L}{m\alpha^2}\left(1+\epsilon\cos\beta\right)^2 \quad -(3)$$

and

$$t = \frac{m\alpha^2}{L}\int\frac{d\beta}{(1+\epsilon\cos\beta)^2} \quad -(4)$$

The time to complete an orbit is:

$$\tau = 2\pi\left(\frac{m\alpha^2}{L}\right) \quad -(5)$$

So

$$t = \frac{\tau}{2\pi}\int_0^\beta\frac{d\beta}{(1+\epsilon\cos\beta)^2} \quad -(6)$$

$$= \frac{\tau}{2\pi}\left[2\tan^{-1}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}\tan\frac{\beta}{2}\right) - \frac{\epsilon(1-\epsilon^2)^{1/2}\sin\beta}{1+\epsilon\cos\beta}\right] \quad -(7)$$

2) It is known that Eq. (7) can be inverted to give the power series in ϵ:

$$\beta(t) = 2\pi\frac{t}{\tau} + 2\epsilon\sin\left(2\pi\frac{t}{\tau}\right) + \frac{5}{4}\epsilon^2\sin\left(4\pi\frac{t}{\tau}\right) + \frac{1}{12}\epsilon^3\left(13\sin\left(6\pi\frac{t}{\tau}\right) - 3\sin\left(2\pi\frac{t}{\tau}\right)\right) + \ldots \quad -(8)$$

It may be possible to use computer to find sufficient terms of eq. (8) so that it can be used for any $0 < \epsilon < 1$ for the ellipse.

The alternative method is Kepler's construction, (Marion and Thornton pp. 263 ff, third edition). This gives:

$$t = \frac{\tau}{2\pi}\left(\phi - \epsilon\sin\phi\right) \quad -(9)$$

where

$$\tan\frac{\beta}{2} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/2}\tan\frac{\phi}{2} \quad -(10)$$

The quantity $2\pi t/\tau$ is the mean anomaly. Computer algebra must be used to find ϕ in terms of t from eq. (9).

We then use the equations:

3)

$$\tan\phi = \frac{L_Z}{L}\tan\beta \quad -(11)$$

and

$$\cos\theta = \left(1 - \left(\frac{L_Z}{L}\right)^2\right)^{1/2} \sin\beta \quad -(12)$$

Therefore $\phi(t)$ and $\theta(t)$ can be found from eqs. (8), (11) and (12).

The Circle

In this case:

$$\epsilon = 0 \quad -(13)$$

so

$$r = \alpha \quad -(14)$$

From eq. for the circle:

$$\beta(t) = 2\pi\frac{t}{\tau} \quad -(15)$$

so:

$$\phi(t) = \tan^{-1}\left(\frac{L_Z}{L}\tan\left(\frac{Lt}{m\alpha^2}\right)\right) \quad -(16)$$

and

$$\theta(t) = \cos^{-1}\left[\left(1 - \left(\frac{L_Z}{L}\right)^2\right)^{1/2} \sin\left(\frac{Lt}{m\alpha^2}\right)\right] \quad -(17)$$

These results apply to types (13) to (16) orbits of Note 275(3): ellipsoidal, one sheet hyperboloidal and ellipsoidal parabolic.