# 273(4): The Binet Equation in 3D, and 3D Orbits in the Solar System and Whirlpool Galaxies

Consider the hamiltonian in three dimensions:

$$H = \frac{1}{2} m v^2 + U(r) \quad -(1)$$

where:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 \quad -(2)$$

If:

$$U(r) = -\frac{k}{r} \quad -(3)$$

then

$$r = \frac{\alpha}{1 + \epsilon \cos\phi} \quad -(4)$$

The force law for this system is given by the three dimensional Binet equation:

$$F(r) = -\frac{L^2}{m r^2} \left( \frac{d^2}{d\phi^2} \left(\frac{1}{r}\right) + \frac{1}{r} \right) \quad -(5)$$

$$= m\ddot{r} - \frac{L^2}{m r^3} = -\frac{k}{r^2}$$

i.e.

$$m\ddot{r} = -\frac{k}{r^2} + \frac{L^2}{m r^3} \quad -(6)$$

). If
$$u(r) = -\frac{L^2}{mr^3} \quad -(7)$$
then:
$$r = \frac{r_0}{\beta} \quad -(8)$$
In 2D theory:
$$\beta = \phi \quad -(9)$$
and eq. (4) becomes a conic section. Eq. (8) becomes a hyperbolic spiral as observed in whirlpool galaxies.

In 3D theory:
$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi\right)^{1/2}} \quad -(10)$$
and
$$\sin\beta = -\frac{L\cos\theta}{\left(L^2 - L_Z^2\right)^{1/2}} \quad -(11)$$
It follows that:
$$\beta = \cos^{-1}\left[\frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi\right)^{1/2}}\right] \quad -(12)$$

and

$$\beta = -\sin^{-1}\left(\frac{L_z\cos\theta}{(L^2-L_z^2)^{1/2}}\right) \quad -(13)$$

Adding eqs. (12) and (13):

$$\beta = \frac{1}{2}\left[\cos^{-1}\left(\frac{\cos\phi}{\left(\cos^2\phi+\left(\frac{L_z}{L}\right)^2\sin^2\phi\right)^{1/2}}\right)\right.$$

$$\left.-\sin^{-1}\left(\frac{L\cos\theta}{(L^2-L_z^2)^{1/2}}\right)\right] \quad -(14)$$

Graphics

1) The three dimensional conic section can be graphed as $r(\phi, \theta)$ using eqs. (4) and (14).

2) The three dimensional hyperbolic spiral can be graphed using eqs (8) and (14).

These give 3D solar system orbits and 3D galaxies.