273(1): The Effect of Three Dimensional Orbit Theory on Kepler's Laws.

The orbit in three dimensional theory is defined by:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\beta\right) \quad -(1)$$

where:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2\sin^2\phi\right)^{1/2}} \quad -(2)$$

This reduces to the orbit of conventional theory if and only if:

$$L \rightarrow L_Z, \quad -(3)$$

in which case:

$$\beta = \phi. \quad -(4)$$

Here α is the half right latitude:

$$\alpha = \frac{L^2}{mk} \quad -(5)$$

and ϵ is the eccentricity:

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2}. \quad -(6)$$

Here L is the magnitude of the total angular momentum, E is the total energy, m is a mass that orbits a mass M in three dimensions. The

2) force of attraction between m and M is:

$$\underline{F} = -\frac{k}{r^2}\underline{e}_r \quad -(7)$$

where r is the separation between m and M. In gravitational theory:

$$k = mMG \quad -(8)$$

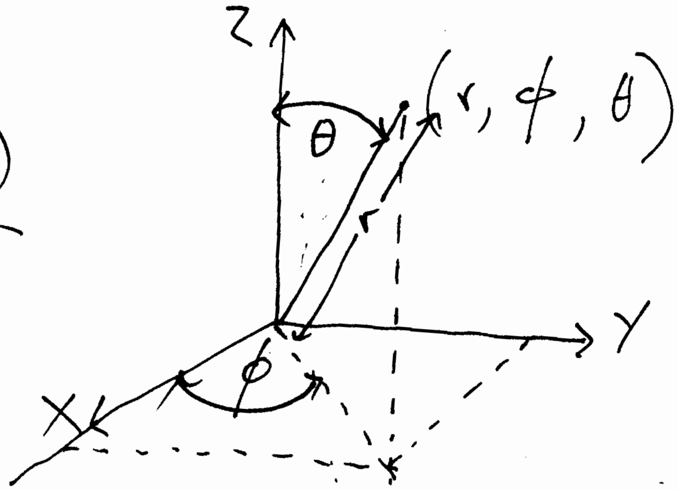
where G is Newton's constant. In electrodynamics:

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(9)$$

where e is the charge on the proton and ϵ_0 the vacuum permittivity. In the spherical polar coordinate system the unit vector $\underline{e}_r$ is defined as:

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \quad -(10)$$

Fig(1)



The spherical polar coordinate system is defined

3) in Fig. (1). In this system the velocity is defined by:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(11)$$

where $\underline{e}_r$ is defined by eq. (10) and where:

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}, \quad -(12)$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(13)$$

so:

$$v^2 = \dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \quad -(14)$$

The angle β is defined by:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta. \quad -(15)$$

The hamiltonian is:

$$H = \frac{1}{2}mv^2 - \frac{k}{r} \quad -(16)$$

and the lagrangian is:

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{r} \quad -(17)$$

Therefore:

$$H = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) - \frac{k}{r} \quad -(18)$$

and the lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) + \frac{k}{r} \quad -(19)$$

The hamiltonian is equal to the total energy:

$$H = E \quad -(20)$$

Therefore:

$$E = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\beta}{dt} \right)^2 \right) - \frac{k}{r} \quad -(21)$$

Eq. (21) may be rewritten as the beta ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(22)$$

where α and ϵ are defined in Eqs. (5) and (6).

Proof

Write:

$$\frac{dr}{dt} = \frac{dr}{d\beta} \frac{d\beta}{dt} \quad -(23)$$

Then:

$$E = \frac{1}{2} m \left(\frac{d\beta}{dt} \right)^2 \left(\left(\frac{dr}{d\beta} \right)^2 + r^2 \right) - \frac{k}{r} \quad -(24)$$

From the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}} \right) \quad -(25)$$

with Lagrangian (19):

$$\frac{d\beta}{dt} = \frac{L}{m r^2} \quad -(26)$$

From eqs. (24) and (26):

5)

$$E = \frac{1}{2}\frac{L^2}{mr^4}\left(\left(\frac{dr}{d\beta}\right)^2 + r^2\right) - \frac{k}{r} \quad -(27)$$

So:

$$\left(\frac{dr}{d\beta}\right)^2 = \frac{2mr^4}{L^2}\left(E - \frac{1}{2}\frac{L^2}{mr^2} + \frac{k}{r}\right)$$

$$= \left(\frac{2mE}{L^2}\right)r^4 - r^2 + \left(\frac{2mk}{L^2}\right)r^3 \quad -(28)$$

Frm eq. (1):

$$\frac{dr}{d\beta} = \frac{\epsilon}{\alpha} r^2 \sin\beta \quad -(29)$$

and

$$\cos^2\beta = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(30)$$

$$= 1 - \sin^2\beta$$

So

$$\left(\frac{dr}{d\beta}\right)^2 = \frac{\epsilon^2}{\alpha^2} r^4 \sin^2\beta$$

$$= \frac{\epsilon^2 r^4}{\alpha^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)$$

$$= \frac{r^4}{\alpha^2}\left(\epsilon^2 - 1\right) - r^2 + \frac{2r^3}{\alpha} \quad -(31)$$

Frm eq. (6):

6)
$$\frac{\epsilon^2 - 1}{\alpha^2} = \frac{2mE}{L^2} \quad -(32)$$

and from eq. (5):
$$\alpha = \frac{L^2}{mk} \quad -(33)$$

So eqs. (28) and (31) are the same, QED. The Lamiltonian (18) is the same as the Beta ellipse (1).

It follows that:
$$m\ddot{r} = -\frac{k}{r^2} + \frac{L^2}{mr^3} = -\frac{L^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) \quad -(34)$$

from the Binet equation:
$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(35)$$

Changes to Kepler's First Law

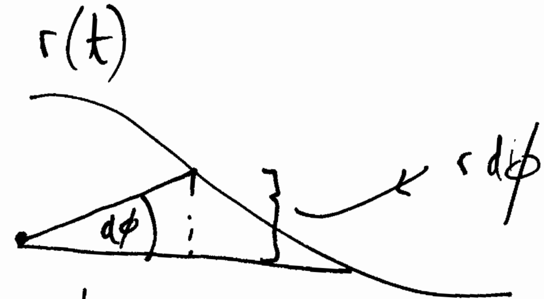
From Eqs. (1) and (2) the orbit is no longer an ellipse in ϕ. It can become completely different from an ellipse. Kepler's First Law states that r is an ellipse in ϕ

7) <u>Changes</u> to <u>Kepler's</u> <u>Second</u> <u>Law</u>

The original second law of Kepler states that

$$\frac{dA}{dt} = \text{constant} \quad -(36)$$

Fig. (2)



Since r is a function of ϕ as in Eqs (1) and (2):

$$dA = \frac{1}{2} r^2 \, d\phi \quad -(37)$$

from Fig (2).

$$\text{So} \quad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\beta}{dt} \frac{d\phi}{d\beta} \quad -(38)$$

From previous work:

$$\frac{d\phi}{d\beta} = \frac{L_Z}{L \sin^2 \theta} \quad -(39)$$

So <u>dA/dt</u> is <u>no longer</u> con<u>stant</u>. From previous work:

$$\sin^2\theta = \left(\frac{L_Z}{L}\right)^2 + \left(1 - \left(\frac{L_Z}{L}\right)^2\right) \left(\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi} \right)$$

$-(40)$

8) So:

$$\frac{dA}{dt} = \frac{L_z}{2m}\left(\left(\frac{L_z}{L}\right)^2 + \left(1-\left(\frac{L_z}{L}\right)^2\right)\left(\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2\sin^2\phi}\right)\right)^{-1}$$

$$\rightarrow \frac{L_z}{2m} \qquad -(41)$$

as

$$L \rightarrow L_z \qquad -(42)$$

and

$$\theta \rightarrow \frac{\pi}{2}, \quad \frac{d\theta}{dt} \rightarrow 0 \qquad -(43).$$

Kepler's Third Law will be dealt with in the next note.