# 273(2): Change to Kepler's Third Law

Kepler's third law is derived conventionally using the method outlined in Note 273(1). Conventionally, in 2D orbital theory:

$$\frac{dA}{dt} = \frac{L_Z}{2m} \quad -(1)$$

so:

$$dt = \left(\frac{2m}{L_Z}\right) dA \quad -(2)$$

This equation is integrated to give:

$$\tau = \int_0^\tau dt = \frac{2m}{L_Z}\int_0^A dA = \frac{2m}{L_Z} A. \quad -(3)$$

The time $\tau$ needed for one orbit is proportional to the area $A$ of the orbit. For an ellipse:

$$A = \pi a b \quad -(4)$$

where $a$ and $b$ are the major and minor semi axes.

Therefore:

$$\tau = \frac{2m\pi ab}{L_Z} \quad -(5)$$

and

$$\tau^2 = \frac{4m^2\pi^2}{L_Z^2} a^2 b^2 \quad -(6)$$

However,

$$b^2 = \alpha a \quad -(7)$$

2)

where $a$ is the semi major axis:

$$a = \frac{L_z^2}{mk} \quad - (8)$$

So

$$\tau^2 = \left(\frac{4m\pi^2}{k}\right) a^3 \quad - (9)$$

This is the conventional form of Kepler's third law.

In 3-D orbital theory, as in note 274(1):

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\beta}{dt} \frac{d\phi}{d\beta} \quad - (10)$$

where:

$$\frac{d\beta}{dt} = \frac{L}{mr^2}, \quad - (11)$$

$$\frac{d\phi}{d\beta} = \frac{L_z}{L \sin^2\theta} \quad - (12)$$

So

$$\frac{dA}{dt} = \frac{L_z}{2m\sin^2\theta} \quad - (13)$$

and

$$dt = \left(\frac{2m\sin^2\theta}{L_z}\right) dA \quad - (14)$$

In order to integrate this equation it must

3) be determined whether A has any dependence on $\theta$. In order to do this note that the original beta ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(15)$$

where:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2\sin^2\phi\right)^{1/2}} \quad -(16)$$

The perihelion of the beta ellipse is defined by the distance of closest approach:

$$r_{min} = \frac{\alpha}{1+\epsilon} \quad -(17)$$

$$= a(1-\epsilon)$$

So at the perihelion:

$$\cos\beta = 1, \quad \beta = 0 \quad -(18)$$

From eq. (16): $\quad \phi = 0 \quad -(19)$

The angle $\theta$ is defined by:

$$\sin^2\theta = \left(\frac{L_Z}{L}\right)^2 + \left(1-\left(\frac{L_Z}{L}\right)^2\right)\left(\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_Z}{L}\right)^2\sin^2\phi}\right) \quad -(20)$$

4) So at the perihelion:

$$\sin^2\theta = 1, \; \theta = \pi/2 \quad -(21)$$

Therefore at the perihelion:

$$\beta = \phi = 0, \quad \theta = \pi/2, \quad -(22)$$

$$r = r_{min}, \quad -(23)$$

$$a = \frac{r_{min}}{1-\epsilon}, \quad -(24)$$

$$b^2 = \alpha a \quad -(25)$$

It follows that the area:

$$A = \pi a b \quad -(26)$$

is defined by fixed angle in Eq. (22). So $A$ has no dependence on variable $\theta$.

Similarly at the aphelion:

$$r = r_{max} = \frac{\alpha}{1-\epsilon} = a(1+\epsilon), \quad -(27)$$

$$\phi = \beta = \pi, \quad \theta = \pi/2 \quad -(28)$$

and $a$ and $b$ are again determined by fixed angles.

Therefore eq. (14) can be integrated

5) as follows:

$$\tau = \int_0^\tau dt = \frac{2m \sin^2\theta}{L_z} \int dA$$

$$= \frac{2mA}{L_z} \sin^2\theta \quad -(29)$$

The area of the beta ellipse is:

$$A = \pi a b \quad -(30)$$

where:

$$b^2 = \alpha a \quad -(31)$$

so

$$\tau^2 = \left( \frac{4m^2\pi^2a^2b^2}{L_z^2} \right) \sin^4\theta \quad -(32)$$

$$= \frac{4m^2\pi^2 \alpha}{L_z^2} a^3 \sin^4\theta$$

where

$$\alpha = \frac{L^2}{mk} \quad -(33)$$

so

$$\boxed{\tau^2 = \left( \frac{4m\pi^2}{k} \left( \frac{L}{L_z} \right)^2 \sin^4\theta \right) a^3} \quad -(34)$$

<u>This is Kepler's third law in 3D theory.</u>

b) In eq. (34):

$$\sin^4\theta = \left[\left(\frac{L_z}{L}\right)^2 + \left(1-\left(\frac{L_z}{L}\right)^2\right)\left(\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi}\right)\right]^2 \quad -(35)$$

Therefore $\tau^2$ is no longer proportional to $a^3$ and develops a dependence on $\phi$.

Graphical Work

It would be very interesting to graph the 3D Kepler laws.