272(4): Conservation of Angular Momentum for All θ.

In three dimensional orbits:

$$\underline{L} = \underline{r} \times \underline{p} \qquad -(1)$$

in the usual notation. The conservation of angular momentum means that:

$$\frac{d\underline{L}}{dt} = \underline{0} \qquad -(2)$$

where

$$\underline{p} = \frac{d\underline{r}}{dt}. \qquad -(3)$$

Therefore:

$$\frac{d\underline{L}}{dt} = \dot{\underline{r}} \times \underline{p} + \underline{r} \times \dot{\underline{p}}.$$

$$= \underline{r} \times \underline{F} = 0 \qquad -(4)$$

where

$$\underline{F} = \dot{\underline{p}} \qquad -(5)$$

$$= m\underline{a}$$

In general: $\qquad -(6)$

$$\underline{a} = \ddot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \dot{\underline{\omega}} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r$$

However:

$$\dot{\underline{\omega}} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r = \underline{0}$$

$$-(7)$$

Therefore:

$$\underline{a} = \ddot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(8)$$

$$= \left(\ddot{r} - \left(r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right)\right)\underline{e}_r$$

and

$$\underline{r} = r\,\underline{e}_r \quad -(9)$$

so

$$\underline{r} \times \underline{F} = 0 \quad -(10)$$

QED.

Conclusion

Angular momentum is conserved for all θ, not just for $\theta = \pi / 2$. This is true for any force law directed in $\underline{e}_r$. The angular momentum $\underline{L}$ is not confined to the Z axis, and both r and ϕ are three dimensional. In general the function of r versus ϕ is not an ellipse, and Kepler's second law is not true.

The usual theory of orbits asserts without proof that $\dot{\theta}$ in eq. (8) is zero and that θ is always $\pi / 2$.