272(3): Force Law and Kepler's Second Law in 3-D Orbits

In 3-D orbits the ellipse is defined by:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\beta\right) \qquad -(1)$$

$$= \frac{1}{\alpha}\left(1 + \epsilon\frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2\sin^2\phi\right)^{1/2}}\right)$$

so the force law is defined by:

$$m\ddot{r} = -\frac{k}{r^2} + \frac{L^2}{mr^3} \qquad -(2)$$

$$= -\frac{L^2}{mr^2}\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right)$$

from the Binet equation:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) - (3)$$

From eq. (1):

$$\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\beta \quad -(4)$$

so the following is true:

2)

$$m\ddot{r} = \frac{\epsilon L^2}{md^3} \cos\beta \left(1+\epsilon\cos\beta\right)^2 \quad -(5)$$

where:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi\right)^{1/2}} \quad -(6)$$

The same result for the planar orbit is:

$$m\ddot{r} = \frac{\epsilon L^2}{md^3} \cos\phi \left(1+\epsilon\cos\phi\right)^2 -(7)$$

so eqns (5) and (7) can be compared directly in graphics.

In 3-D orbits Kepler's second law of equal areas in equal times is changed.

This is because:

$$L = mr^2 \frac{d\beta}{dt} \quad -(8)$$

and

$$dA = \frac{1}{2} r^2 \, d\phi \quad -(9)$$

so

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\beta}{dt} \frac{d\phi}{d\beta} \quad -(10)$$

where:

3)

$$\frac{d\phi}{dp} = \frac{L_z}{L\sin^2\theta} \quad -(11)$$

from previous work. So dA/dt is no longer constant, QED.

Eq. (10) implies ~~that~~:

$$\frac{dA}{dt} = \frac{L_z}{2m\sin^2\theta} \quad -(12)$$

From previous work:

$$1 - \left(\frac{L^2}{L^2 - L_z^2}\right)\cos^2\theta = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2\sin^2\phi} \quad -(13)$$

so:

$$\sin^2\theta = \left(\frac{L_z}{L}\right)^2 + \left(1 - \left(\frac{L_z}{L}\right)^2\right)\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2\sin^2\phi} \quad -(14)$$

and Kepler's Second Law becomes:

$$\frac{dA}{dt} = \frac{L_z}{2m}\left(\left(\frac{L_z}{L}\right)^2 + \left(1 - \left(\frac{L_z}{L}\right)^2\right)\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2\sin^2\phi}\right)^{-1} \quad -(15)$$

QED. Eq. (15) can be graphed.