268(2): Derivation of the Ellipse from the Classical Hamiltonian and Results from the Quantum Theory

The classical hamiltonian is:

$$H = E = \frac{p^2}{2m} - \frac{k}{r} \quad -(1)$$

as in previous notation. It can be expressed as:

$$E = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{k}{r} \quad -(2)$$

so

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2mr^4}{L^2}\left(E + \frac{k}{r}\right) - r^2 \quad -(3)$$

using:

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(4)$$

and

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(5)$$

Now consider the ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(6)$$

It follows that:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\epsilon^2 r^4}{\alpha^2}\left(1-\cos^2\theta\right) \quad -(7)$$

where

$$\cos^2\theta = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(8)$$

After some algebra:

2)

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2}{\alpha^2}\left(r^2(\epsilon^2-1) + 2\alpha r\right) - r^2 \quad -(9)$$

Comparing Eqs (3) and (9):

$$\boxed{\alpha = \frac{L^2}{mk}} \quad -(10)$$

and

$$\frac{\epsilon^2-1}{\alpha^2} = \frac{2mE}{L^2} \quad -(11)$$

so

$$\epsilon^2 = 1 + \frac{2mE\alpha^2}{L^2}$$

$$\boxed{\epsilon^2 = 1 + \frac{2EL^2}{mk^2}} \quad -(12)$$

It is seen that the ellipse (6) is a direct re-expression of the fundamental classical hamiltonian (1).

This is true s.t. for gravitation and electrodynamics. For gravitation:

$$k = mMG \quad -(13)$$

and for electrodynamics:

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(14)$$

3) Note that the Bohr theory of the atom is obtained directly by using:

$$L = n\hbar \quad -(15)$$

and

$$\epsilon = 0. \quad -(16)$$

In the Schroedinger theory the following results have been computed in recent work:

$$\text{1)} \quad \langle r \rangle = \left\langle \frac{\alpha}{1+\epsilon\cos\phi} \right\rangle = \frac{\alpha}{(1-\epsilon^2)^{1/2}} \quad -(17)$$

for all orbitals of H.

$$\text{2)} \quad \langle r \rangle = \left\langle \frac{\alpha}{1+\epsilon\cos\theta} \right\rangle$$ is different for every orbital of H.

$$\text{3)} \quad \langle r \rangle = \left\langle \frac{\alpha}{1+\epsilon\cos(x\phi)} \right\rangle$$ is different for all orbitals of H.

$$\text{4)} \quad \langle r \rangle = \left\langle \frac{\alpha}{1+\epsilon\cos(x\theta)} \right\rangle$$ is different for all orbitals of H.

$$\text{5)} \quad \left\langle \frac{1}{r} \right\rangle = \left\langle \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right) \right\rangle \quad -(18)$$

$$= \left\langle \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) \right\rangle$$

$$= \frac{1}{\alpha}$$

for all orbitals of H.

4)

$$\langle \cos\theta \rangle = \langle \cos\phi \rangle = 0 \quad -(19)$$

for all orbitals of H.

<u>Interesting Computations</u>

1) $\langle \cos(x\theta) \rangle$, $\langle \cos(x\phi) \rangle$ for each orbital.

2) $\langle \cos(n\theta) \rangle$, $\langle \cos(n\phi) \rangle$ for each orbital.

The first set corresponds to relativistic correction due to x, and the second set to expectation values for Eckardt quantization.
