266(8): Bohr Theory of the Atom and Relativistic Corrections

The Bohr theory of the atom is described by the force law:

$$m\frac{d^2r}{dt^2} = -\frac{e^2}{4\pi\epsilon_0 r^2} + \frac{L^2}{mr^3} = 0. \quad -(1)$$

It is a theory of circular orbits and

$$\frac{dr}{dt} = 0 \quad -(2)$$

So:

$$\frac{L^2}{mr^3} = \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(3)$$

and the Bohr radius is:

$$r = \frac{4\pi\epsilon_0 L^2}{me^2}. \quad -(4)$$

The Bohr quantization is:

$$L = n\hbar, \quad -(5)$$

where n is now known as the principal quantum number.

The total velocity of the electron in the Bohr atom is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(6)$$

2) In view of eq. (2):

$$V = r\omega. \quad -(7)$$

where

$$\omega = \frac{d\theta}{dt} = \frac{L}{mr^2}, \quad -(8)$$

so

$$\boxed{V = V_\theta = \frac{L}{mr} = \frac{n\hbar}{mr}} \quad -(9)$$

where r is the Bohr radius (4).

Note carefully that this same velocity is used in the Lorentz factor of the Bohr Sommerfeld theory of the atom.

The hamiltonian of the Bohr atom is:

$$H = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(10)$$

$$= T + V$$

in which

$$v = v_\theta = \frac{L}{mr} \quad -(11)$$

So

$$H = E = \frac{L^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r} \quad -(12)$$

From eq. (3):

$$\frac{L^2}{2mr^2} = \frac{e^2}{8\pi\epsilon_0 r} \quad -(13)$$

So:

$$E = \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r}$$

$$E = -\frac{e^2}{8\pi\epsilon_0 r} \qquad -(14)$$

The energy is negative valued. This is because the electron is attracted to the proton. From eqs. (4) and (14):

$$\boxed{E = -\frac{me^4}{32\pi^2\epsilon_0^2 n^2 \hbar^2}} \qquad -(15)$$

where:

$$n = 1, 2, 3, \ldots \qquad -(16)$$

These are the non-relativistic energy levels of the hydrogen atom as observed in the main details of atomic spectra.

This theory is precisely analogous to the Hooke / Newton / Leibniz theory of orbits:

$$m\frac{d^2 r}{dt^2} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} \qquad -(17)$$

(G. von Leibniz, 1689).

One theory is transformed into another

4) using:

$$k = mMG \rightarrow \frac{e^2}{4\pi\epsilon_0} \quad -(18)$$

In ECE theory both are examples of Cartan geometry with spin connection $\omega = d\theta / dt$.

The orbit of both theories is a conical section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(19)$$

where the half right latitude is:

$$\alpha = \frac{L^2}{mk} \quad -(20)$$

and the eccentricity is:

$$\epsilon = \left(1 + \frac{2|E|L^2}{mk^2}\right)^{1/2} \quad -(21)$$

the semi major axis is:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|} \quad -(22)$$

for the ellipse and the semi minor axis for the ellipse is

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2m|E|)^{1/2}} \quad -(23)$$

The distance of closest approach (the perihelion) is defined by:

5)

$$r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \quad -(24)$$

and the aphelion, the distance of maximum separation, is given by:

$$r_{max} = a(1+\epsilon) = \frac{\alpha}{1-\epsilon} \quad -(25)$$

For the ellipse:

$$0 < \epsilon < 1. \quad -(26)$$

The Bohr radius is given by the half right latitude:

$$\alpha = r = \frac{L^2}{mk} = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \quad -(27)$$

The Bohr theory makes the assumption that the ellipse reduces to a circle, so:

$$\epsilon = 0. \quad -(28)$$

From eq. (21) this means that:

$$\frac{2|E|L^2}{mk^2} = -1 \quad -(29)$$

so

$$|E| = -\frac{me^4}{32\pi^2\epsilon_0^2 n^2 \hbar^2} \quad -(30)$$

which is eq. (15), Q.E.D.

6) The relativistic Bohr Sommerfeld theory is based on the hamiltonian:

$$H = E = (\gamma - 1) mc^2 - \frac{k}{r} \quad -(31)$$

in which the relativistic kinetic energy is:

$$T = (\gamma - 1) mc^2 \quad -(32)$$

and in which the Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(33)$$

The Lorentz factor derives from the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = (c^2 - v^2) dt^2 \quad -(34)$$

so

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(35)$$

Note carefully that v is the same for the non-relativistic and relativistic theories and is defined by:

$$c^2 d\tau^2 = (c^2 - v^2) dt^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(36)$$

so

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \quad -(37)$$

which is eq. (6), Q.E.D.

7) Therefore in order to define the relativistic correction of the Bohr theory, one must use the Bohr linear velocity (9) in Eq. (31). So the relativistically corrected energy levels of the H atom are:

$$E = \left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(38)$$

in which:

$$v = \frac{n\hbar}{mr} \quad -(39)$$

and:

$$r = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2} \quad -(40)$$

Defining the fine structure constant as:

$$\alpha_f = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(41)$$

it is found that the Bohr velocity is:

$$\boxed{v = \frac{\alpha_f}{n} c} \quad -(42)$$

and that the Bohr radius is:

$$\frac{1}{r} = \left(\frac{mc}{\hbar}\right)\frac{\alpha_f}{n^2} \quad -(43)$$

where the Compton / de Broglie wavelength is:

8)

$$\lambda = \frac{h}{mc} = 2\pi \frac{\hbar}{mc} \quad -(44)$$

So

$$\frac{mc}{\hbar} = \frac{2\pi}{\lambda_c} \quad -(45)$$

and

$$\boxed{\frac{1}{r} = \frac{2\pi}{\lambda_c} \frac{\alpha_f}{n^2}} \quad -(46)$$

Therefore the energy levels from eq. (38) are:

$$E = \left(\left(1 - \frac{\alpha_f^2}{n^2} \right)^{-1/2} - 1 \right) mc^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

$$= \left(\left(1 - \frac{\alpha_f^2}{n^2} \right)^{-1/2} - 1 \right) mc^2 - \frac{me^4}{16\pi\epsilon_0^2 n^2 \hbar^2}$$

$$= \left(\left(1 - \frac{\alpha_f^2}{n^2} \right)^{-1/2} - 1 \right) mc^2 - mc^2 \frac{\alpha_f^2}{n^2} \quad -(47)$$

so

$$\boxed{E = \left(\left(1 - \frac{\alpha_f^2}{n^2} \right)^{-1/2} - 1 - \frac{\alpha_f^2}{n^2} \right) mc^2} \quad -(48)$$

The non-relativistic energy levels are:

$$E = -\frac{mc^2}{2} \frac{\alpha_f^2}{n^2} \quad -(49)$$

) The fine structure constant is:

$$\alpha_f = 0.007297351 \quad -(50)$$

so to an excellent approximation:

$$\left(1-\frac{\alpha_f^2}{n^2}\right)^{-1/2} \sim 1+\frac{1}{2}\frac{\alpha_f^2}{n^2} \quad -(51)$$

and in this approximation eq. (48) reduces to eq. (49), Q.E.D.

This is a simpler solution than that of Sommerfeld and uses the Bohr quantum number n.