266(7): Derivation of the Precessing Ellipse from the Sommerfeld Hamiltonian.

The Sommerfeld Hamiltonian is:

$$H = (\gamma - 1) mc^2 - \frac{k}{r} \quad -(1)$$

where

$$k = \frac{e^2}{4\pi \epsilon_0} \quad -(2).$$

The total energy is defined by

$$H = E, \quad -(3)$$

and the Lorentz factor by:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2. \quad -(5)$$

Define the fine structure constant by:

$$\alpha_f = \frac{e^2}{4\pi \hbar c \epsilon_0}. \quad -(6)$$

Then eq. (1) is:

$$\gamma = 1 + \frac{E}{mc^2} + \frac{\hbar}{mc} \frac{\alpha_f}{r} \quad -(7)$$

The quantization conditions used by Sommerfeld are:

2)

$$\oint p_r\,dr = \oint \frac{h}{\lambda}\,dr = nh, \quad n = 0, 1, 2, \ldots \quad -(8)$$

and

$$\oint p_\theta\,d\theta = \oint mrv\,d\theta = \ell h \quad -(9)$$

So γ in eq. (7) may be expressed in terms of the two quantum numbers n and ℓ and E evaluated by computer algebra. That will be the subject of the next note. Eq. (7) can be rearranged as:

$$\frac{\gamma mc}{\hbar\alpha_f} = \frac{mc}{\hbar\alpha_f}\left(1+\frac{E}{mc^2}\right) + \frac{1}{r} \quad -(10)$$

Multiply both sides by $\alpha_f^2/(c^2L^2)$:

$$\frac{\gamma m\alpha_f}{\hbar cL^2} = \frac{m\alpha_f}{\hbar cL^2}\left(1+\frac{E}{mc^2}\right) + \left(\frac{\alpha_f}{cL}\right)^2\frac{1}{r} \quad -(11)$$

The left hand side is expressed as follows:

$$\frac{\gamma m\alpha_f\hbar c}{L^2} = -\frac{mr^2F(r)\gamma}{L^2} \quad -(12)$$

So

$$F(r) = -\frac{\alpha_f\hbar c}{r^2} = -\frac{e^2}{4\pi\epsilon_0 r^2} \quad -(13)$$

Eq. (12) follows from the multiplication of both sides of eq. (11) by $\hbar^2 c^2$ to give:

$$\frac{\gamma m c\, d_g \hbar}{L^2} = \frac{m c \hbar d_g}{L^2}\left(1 + \frac{E}{mc^2}\right) + \left(\frac{d_g \hbar}{L}\right)^2 \frac{1}{r} \quad -(14)$$

$$= -\frac{m r^2 F(r)}{L^2}\,\gamma$$

The Binet equation is now used:

$$F(r) = -\frac{L^2}{m r^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(15)$$

so

$$\gamma\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = \frac{m c \hbar d_g}{L^2}\left(1 + \frac{E}{mc^2}\right) + \left(\frac{d_g \hbar}{L}\right)^2 \frac{1}{r} \quad -(16)$$

Error One in Wikipedia Article

The factor γ is missing from the left hand side of the Wikipedia entry. One can only guess that it has been left out by error or approximated by unity. The correct expression is:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{m c \hbar d_g}{\gamma L^2}\left(1 + \frac{E}{mc^2}\right) + \left(\frac{d_g^2 \hbar^2}{\gamma L^2}\right) \frac{1}{r} \quad -(17)$$

4)

Eq. (17) can be expressed as:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2}{r} + K \quad -(18)$$

where

$$x^2 = 1 - \frac{1}{\gamma}\left(\frac{\alpha_g^2 \hbar^2}{L^2}\right) \quad -(19)$$

$$K = \frac{mc\hbar\,\alpha_g}{\gamma L^2}\left(1 + \frac{E}{mc^2}\right) \quad -(20)$$

where

$$\frac{1}{\gamma} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad -(21)$$

Eq. (18) implies:

$$\frac{1}{r} = \frac{K}{x^2} + A\cos(x\theta) \quad -(22)$$

Error 2 in the Wikipedia Article

The factor x^2 was left out of eq. (22).

Check

From eq. (22):

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -A x^2 \cos(x\theta) \quad -(23)$$

where

$$\cos(x\theta) = \frac{1}{A}\left(\frac{1}{r} - K\right) \quad -(24)$$

5)

so
$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{Ax^2}{A}\left(\frac{1}{r} - K\right) \quad -(25)$$
$$= -\frac{x^2}{r} + K$$

which is eq. (18), QED.

Finally Eq. (22) can be expressed as the precessing ellipse:
$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos(x\theta)\right) \quad -(26)$$

where
$$\alpha = \frac{x^2}{K} \quad -(27)$$

and
$$\epsilon = \frac{x^2 A}{K} = \alpha A \quad -(28)$$

Without loss of generality we can use:
$$A = \frac{\epsilon}{\alpha} \quad -(29)$$

Approximation

In general:
$$x^2 = 1 - \left(1 - \frac{v^2}{c^2}\right)^{1/2}\left(\frac{d_g \mathcal{L}}{L}\right)^2 \quad -(30)$$

and:

6)

$$\kappa = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \frac{mc\hbar\alpha_f}{L^2} \left(1 + \frac{E}{mc^2}\right) \quad -(31)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(32)$$

$$= \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2}$$

Sommerfeld Quantization

This is $L = n_\theta \hbar$ $-(33)$

and the linear momentum is quantized as:

$$p = \hbar\kappa \quad -(34)$$

Therefore the velocity can be expressed as:

$$v^2 = \frac{\hbar^2}{m^2}\kappa^2 + \frac{\hbar^2 n_\theta^2}{m^2 r^2}$$

$$= \frac{\hbar^2}{m^2}\left(\kappa^2 + \frac{n_\theta^2}{r^2}\right) \quad -(35)$$

Finally assume:

$$\kappa^2 = \frac{n_r^2}{r^2} \quad -(36)$$

so

$$\boxed{v^2 = \frac{\hbar^2}{m^2 r^2}\left(n_r^2 + n_\theta^2\right)} \quad -(37)$$

7)

Note that the Bohr model is:

$$E = -\frac{k}{r} + \frac{1}{2}mv^2$$

$$= \frac{me^4}{8\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(38)$$

with

$$L = n\hbar \quad -(39)$$

The relativistic correction to eq. (38) is therefore:

$$E = -\frac{k}{r} + \frac{1}{2}mv^2 + (\gamma - 1)mc^2 - \frac{1}{2}mv^2$$

$$= \frac{me^4}{8\pi^2 \epsilon_0^2 \hbar^2 n^2} + (\gamma - 1)mc^2 - \frac{1}{2}mv^2 \quad -(40)$$

$$= \frac{me^4}{8\pi^2 \epsilon_0^2 \hbar^2 n^2} + \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 - \frac{1}{2}mv^2 \quad -(41)$$

with

$$v^2 = \frac{\hbar^2}{mr^2}\left(n_r^2 + n_\theta^2\right) \quad -(42)$$

At the Bohr radius:

$$r = \frac{4\pi \epsilon_0 \hbar^2}{me^2} \quad -(43)$$