266(3) Bohr Sommerfeld Quantization from x Theory

Consider the infinitesimal line element of x theory:

$$ds^2 = c^2 d\tau^2 = (c^2 - v^2)dt^2 \quad -(1)$$
$$= c^2 dt^2 - dr^2 - r^2 d\theta^2$$

It follows that the orbital velocity is:

$$v^2 = v_r^2 + v_\theta^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(2)$$

and that:

$$mc^2 = mc^2\left(\frac{dt}{d\tau}\right)^2 - m\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 \quad -(3)$$

Rearranging eq. (3) gives the relativistic kinetic energy in the form:

$$\frac{1}{2}m\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) = \frac{1}{2}(\gamma^2 - 1)mc^2 \quad -(4)$$

Defining the potential energy by $U$, the relativistic hamiltonian and lagrangian are:

$$H = \frac{1}{2}m\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) + U = \frac{1}{2}(\gamma^2 - 1)mc^2 + U \quad -(5)$$

and

$$\mathcal{L} = \frac{1}{2}m\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) - U = \frac{1}{2}(\gamma^2 - 1)mc^2 - U \quad -(6)$$

The relativistic velocity is:

2)

$$\underline{v} = \frac{dr}{d\tau} \quad -(7)$$

and the relativistic momentum is:

$$p = \frac{\partial \mathcal{L}}{\partial v} = \gamma m v \quad -(8)$$

The Lorentz factor is:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(9)$$

where $v$ is defined in Eq. (2).

The hamiltonian and lagrangian can be defined as:

$$H = \frac{1}{2}\gamma^2 m v^2 + U = \frac{1}{2}\left(\gamma^2 - 1\right) m c^2 + U \quad -(10)$$

and

$$\mathcal{L} = \frac{1}{2}\gamma^2 m v^2 - U = \frac{1}{2}\left(\gamma^2 - 1\right) m c^2 - U \quad -(11)$$

The relativistic Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(12)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(13)$$

These give:

$$m\frac{d^2 r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2 = -\frac{\partial U}{\partial r} = F \quad -(14)$$

3) where $F$ is the relativistic force, and

$$L = mr^2 \frac{d\theta}{d\tau} \quad -(15)$$

where $L$ is the relativistic total angular momentum.

Eq. (14) can be reexpressed as:

$$F(r) = -\frac{L^2}{mr^2}\left(\gamma \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(16)$$

So:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)_{rel} = \gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)_{nonrel} \quad -(17)$$

<u>The relativistic force is</u> $\gamma^2$ <u>multiplied by the non relativistic force.</u>

The Bohr Sommerfeld model of the atom is based on the hamiltonian:

$$H = (\gamma - 1)mc^2 - \frac{kZe^2}{r} \quad -(18)$$

in conventional notation.

In the limit $v \ll c$ the definition (5) and (18) give the same result:

$$T = \frac{mc^2}{2}\left(\gamma^2 - 1\right) = \left(\frac{1}{2}\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)mc^2\right)$$

$$\sim \frac{1}{2} mv^2 \quad -(19)$$

and:

4)

$$T = mc^2(\gamma - 1) = mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)$$

$$\sim \frac{1}{2}mv^2 \quad -(20)$$

From eq. (18):

$$\gamma = 1 + \frac{H}{mc^2} + \frac{kZe^2}{mc^2}\frac{1}{r} \quad -(21)$$

Rearranging eq. (21) gives:

$$\frac{mkZe^2}{L^2}\left(1 + \frac{H}{mc^2}\right) + \frac{mkZe^2}{L^2}\cdot\frac{kZe^2}{mc^2}\cdot\frac{1}{r}$$

$$= \frac{mkZe^2}{L^2}\gamma \quad -(22)$$

i.e

$$\frac{mkZe^2}{L^2}\left(1 + \frac{H}{mc^2}\right) + \frac{k^2Z^2e^4}{c^2L^2}\frac{1}{r}$$

$$= \frac{mkZe^2}{L^2}\gamma \quad -(23)$$

Using Eq. (16), the right hand side of eq. (23) can be expressed as:

$$-\frac{mr^2F(r)}{L^2} = \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}$$

$$= \frac{mkZe^2\gamma}{L^2} \quad -(24)$$

5) It follows that:

$$F(r) = -\gamma \frac{kZe^2}{r^2} \quad -(25)$$

which is the relativistic force, Q.E.D.

Therefore the equation of motion of the Bohr-Sommerfeld atom is:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\left(1 - \frac{k^2Z^2e^4}{c^2L^2}\right)\frac{1}{r} + \frac{mkZe^2}{L^2}\left(1 + \frac{H}{mc^2}\right)$$

$$:= -\frac{\omega_0^2}{r} + K \quad -(26)$$

where

$$\omega_0^2 = 1 - \frac{k^2Z^2e^4}{c^2L^2} \quad -(27)$$

and

$$K = \frac{mkZe^2}{L^2}\left(1 + \frac{H}{mc^2}\right) \quad -(28)$$

A solution of Eq. (26) is:

$$\frac{1}{r} = K + A\cos(\omega_0\theta) \quad -(29)$$

which is found by twice integrating:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -A\omega_0^2\cos(\omega_0\theta) \quad -(30)$$

 and adding $K$ as a constant of integration.

Therefore:

$$r = \frac{1}{K + A\cos(\omega_0 \theta)},$$

$$\boxed{r = \frac{1/K}{1 + A\cos(\omega_0 \theta)}} \quad -(31)$$

This is a precessing ellipse of x theory, where

$$x = \omega_0$$
$$= \left(1 - \frac{k^2 Z^2 e^4}{c^2 L^2}\right)^{1/2} \quad -(32)$$

The half right latitude is:

$$\alpha = 1/K \quad -(33)$$

and the eccentricity is:

$$\epsilon = A \quad -(34)$$

The quantum conditions are defined by:

$$\oint L \, d\theta = 2\pi L = n_L h \quad -(35)$$

and

$$\oint p_r \, dr = L \oint \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 d\theta = n_r h \quad -(36)$$

This procedure may be repeated for

) gravitation as follows:

$$H = (\gamma - 1) mc^2 - \frac{mMG}{r}, \quad -(37)$$

so

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\left(1 - \frac{m^2 M^2 G^2}{c^2 L^2}\right)\frac{1}{r} + \frac{m^2 MG}{L^2}\left(1 + \frac{H}{mc^2}\right) \quad -(38)$$

Therefore:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(39)$$

where

$$\alpha = \frac{1}{k} = \frac{L^2}{m^2 MG}\left(1 + \frac{H}{mc^2}\right)^{-1}. \quad -(40)$$

and

$$A = \epsilon. \quad -(41)$$

The $x$ factor is:

$$x = \omega_0 = \left(1 - \frac{m^2 M^2 G^2}{L^2}\right)^{1/2} \quad -(42)$$

Therefore $x$ theory as defined in eqs. (39) to (42) produces quantization. For example the well known petal orbitals of the Bohr Sommerfeld theory emerge from eq. (39)