266(2): Double Check on the Equivalence Principle for V_θ

As shown by a Lagrangian analysis for example, the force is defined by:

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \quad -(1)$$

which is Eq. (7.18) of Marion and Thornton, "Classical Dynamics", 3rd ed., (Harcourt, 1988)

The kinetic energy is:

$$T = \int F(r)\,dr \quad -(2)$$

$$= -U = \frac{mMG}{r}$$

The total linear velocity of the orbit is:

$$\underline{v} = \underline{v}_r + \underline{\omega} \times \underline{r} \quad -(3)$$

$$= \frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta$$

where

$$\underline{\omega} = \frac{d\theta}{dt}\underline{k} \quad -(4)$$

The total kinetic energy is:

$$T = \frac{1}{2} m \underline{v} \cdot \underline{v}. \quad -(5)$$

The Thomas velocity is defined by:

2)

$$\underline{v}_\theta = \underline{\omega} \times \underline{r} = \omega r \, \underline{e}_\theta \quad -(6)$$

so

$$v_\theta = |\underline{v}_\theta| = \omega r, \quad -(7)$$

and

$$v_r = 0 \quad -(8).$$

Therefore

$$\underline{v} = \underline{v}_\theta \quad -(8)$$

and the total kinetic energy is:

$$T = \frac{1}{2} m v_\theta^2 = m\frac{MG}{r} \quad -(9)$$

and

$$\boxed{v_\theta^2 = \frac{2MG}{r}} \quad -(10)$$

Q.E.D.

The Thomas velocity v_θ defines a rotating frame as follows:

$$d\theta' = d\theta + \frac{v_\theta}{r} dt \quad -(11)$$

$$= d\theta + \omega \, dt$$

where

$$\underline{\omega} = \omega \underline{k} \quad -(12)$$

is a constant. In ECE theory this is a constant spin connection of Cartan geometry, and is the angular velocity of the rotating frame.

3) A circular orbit is defined by

$$V_\theta = \omega r \quad - (13)$$

which is the Thomas velocity. This is the linear orbital velocity of a point rotating in a circle. The orbit of a circle is:

$$r = \alpha \quad - (14)$$

so in Eq. (10):

$$\boxed{V_\theta^2 = \frac{2MG}{\alpha}} \quad - (15)$$

Q.E.D.

As shown in previous notes, e.g. 266(1), eq. (11) implies that:

$$\boxed{\frac{dt}{d\tau} = x = 1 + \frac{3MG}{c^2 \alpha}} \quad - (16)$$

The effect of precession is:

$$\theta' = x\theta = \frac{dt}{d\tau}\theta \quad - (17)$$

so

$$d\theta' = \frac{dt}{d\tau} d\theta \quad - (18)$$

and

$$\boxed{\frac{d\theta'}{dt} = \frac{d\theta}{d\tau}} \quad - (19)$$

4) The turning point of an elliptical orbit is also defined by Eq. (14). The elliptical orbit is defined in the rotating frame, and the entire ellipse rotates with the velocity defined in Eq. (15). This produces the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(20)$$

where

$$x = 1 + \frac{3MG}{c^2 \alpha} \quad -(21)$$

which is the experimental result to very high contemporary precision, Q.E.D. For an ellipse:

$$\alpha = a(1-\epsilon^2), \quad -(22)$$

$$0 < \epsilon < 1$$

For a hyperbola:

$$\alpha = a(\epsilon^2 - 1), \quad -(23)$$

$$\epsilon > 1.$$

Suggested Graphical Work

1) Plot the orbit (20) for increasing M, and fixed a and ϵ, respectively the semi major axis and eccentricity. Investigate the limit $M \to \infty$, $\alpha \to 0$. This is the mythical "black hole" of standard physics.

2) Repeat for the hyperbola.

Eq. (1) can be rewritten as:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right) \quad -(24)$$

using:

$$\omega = \frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(25)$$

where L is the total angular momentum of the system, a constant of motion. Eq. (24) is Eq. (7.21) of Marion and Thornton.

1) For the circular orbit:

$$r = \alpha \quad -(26)$$

so

$$F(r) = -\frac{L^2}{mr^3} \quad -(27)$$

and

$$\frac{d^2 r}{dt^2} = 0. \quad -(28)$$

Eq. (24) can be rewritten as:

$$\frac{d^2 r}{dt^2} = -\frac{L^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = F(r) + \frac{L^2}{mr^3}$$

$$-(29)$$

For a circular orbit:

6)

$$\frac{d^2 r}{dt^2} = -\frac{L^2}{mr^3} + \frac{L^2}{mr^3} = 0. \quad -(30)$$

The centripetal force is:

$$\underline{F}_p = -\frac{L^2}{mr^3}\underline{e}_r - (31)$$

and the centrifugal force is:

$$\underline{F}_F = \frac{L^2}{mr^3}\underline{e}_r. \quad -(32)$$

These are equal and opposite and are radially directed along $\underline{e}_r$, the vector joining m and M.

Note carefully that all these concepts emerge from the equivalence principle, eq. (1), and the conservation of total angular momentum L.

2) For the elliptical orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(33)$$

Eq. (24) gives:

$$F(r) = -\frac{mMG}{r^2} \quad -(34)$$

which is the inverse square law of Hooke and Newton.

7) So
$$m\frac{d^2r}{dt^2} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} \quad -(35)$$

This equation was first inferred by von Leibnitz in 1689. These results emerge from:
$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) \quad -(36)$$

so
$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\theta \quad -(37)$$

where
$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right). \quad -(38)$$

So
$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{\alpha} - \frac{1}{r} \quad -(39)$$

and
$$F(r) = -\frac{L^2}{mr^2}\left(\frac{1}{\alpha} - \frac{1}{r} + \frac{1}{r}\right)$$
$$= -\frac{L^2}{mr^2\alpha}. \quad -(40)$$

Therefore
$$\alpha = \frac{L^2}{m^2MG} \quad -(41)$$

for the elliptical orbit.

8) 3) For the precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(42)$$

it follows that

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos(x\theta)\right) \quad -(43)$$

So

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -x^2\frac{\epsilon}{\alpha}\cos(x\theta) \quad -(44)$$

$$= x^2\left(\frac{1}{\alpha}-\frac{1}{r}\right)$$

Therefore:

$$F(r) = -\frac{L^2}{mr^2}\left(x^2\left(\frac{1}{\alpha}-\frac{1}{r}\right)+\frac{1}{r}\right) \quad -(45)$$

and

$$m\frac{d^2r}{dt^2} = F(r) + \frac{L^2}{mr^3} \quad -(46)$$

$$= -x^2\frac{L^2}{mr^2}\left(\frac{1}{\alpha}-\frac{1}{r}\right)$$

$$\boxed{m\frac{d^2r}{dt^2} = x^2\left(-\frac{mMG}{r^2}+\frac{L^2}{mr^3}\right)} \quad -(47)$$

Note carefully that this is the Leibniz force law (35) multiplied by:

9)
$$x^2 = \left(1 + \frac{3MG}{c^2 \alpha}\right)^2 \quad -(48)$$

This is the rigorously correct force law of the precessing ellipse (42).

The obsolete and incorrect Einstein force law is:

$$m \frac{d^2 r}{dt^2} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} - \frac{3MG}{c^2} \frac{L^2}{mr^4} \quad -(49)$$

If an attempt is made to equate eqns. (47) and (49) the result is an infinity in the Einstein theory, proving conclusively that it is wildly incorrect.

Suggested Graphical Work

Plot Eq. (47) against Eq. (49) for large M and small α:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(50)$$

The mythical Einstein theory is said to produce a "black hole" in this limit.