Note 266(8a): (Checking Eq. (48) of Note 266(8))

The Bohr energy is:

$$E = \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} \quad -(1)$$

where the Bohr radius is:

$$r = \frac{4\pi\epsilon_0 L^2}{me^2} \quad -(2)$$

So

$$E = \frac{me^4}{32\pi^2\epsilon_0^2 L^2} - \frac{me^4}{16\pi^2\epsilon_0^2 L^2} \quad -(3)$$

$$= -\frac{me^4}{32\pi^2\epsilon_0^2 n^2\hbar^2}$$

The fine structure constant is:

$$\alpha_f = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(4)$$

so

$$E = \frac{\alpha_f^2}{n^2} mc^2 \left(\frac{1}{2} - 1\right) = -\frac{1}{2}\frac{\alpha_f^2}{n^2} mc^2 \quad -(5)$$

using

$$L = \hbar n \quad -(6)$$

The Bohr velocity is:

$$v = \omega r = \frac{L}{mr^2} r = \frac{L}{mr} = \frac{n\hbar}{mr} \quad -(7)$$

$$= \frac{n\hbar me^2}{4\pi m\epsilon_0 L^2} = \frac{e^2}{4\pi\epsilon_0\hbar n} = \frac{\alpha_f c}{n}$$

So

$$\frac{v}{c} = \frac{\alpha_f}{n} \quad -(8)$$

2) The Sommerfeld energy levels are:

$$E = (\gamma - 1)mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(9)$$

where

$$\frac{1}{r} = \frac{me^2}{4\pi\epsilon_0 n^2 \hbar^2} \quad -(10)$$

So

$$E = (\gamma-1)mc^2 - \frac{me^4}{16\pi^2\epsilon_0^2 n^2\hbar^2}$$

$$= (\gamma-1)mc^2 - \frac{\alpha_f^2}{n^2} mc^2 \quad -(11)$$

$$= mc^2\left((\gamma-1) - \left(\frac{\alpha_f}{n}\right)^2\right)$$

in which:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(1 - \left(\frac{\alpha_f}{n}\right)^2\right)^{-1/2} \quad -(12)$$

So:

$$E = mc^2\left(\left(1 - \left(\frac{\alpha_f}{n}\right)^2\right)^{-1/2} - 1 - \frac{\alpha_f^2}{n^2}\right) \quad -(13)$$

which is Eq. (48), Q.E.D.

To an excellent approximation:

$$E = mc^2\left(1 + \frac{1}{2}\left(\frac{\alpha_f}{n}\right)^2 - 1 - \frac{\alpha_f^2}{n^2}\right)$$

$$= -\frac{1}{2}mc^2\left(\frac{\alpha_f}{n}\right)^2 \quad -(14)$$

which is Eq. (5) QED