266(9): Self Consistent Calculation of the Sommerfeld Theory

In addition to the total energy:

$$H = E = (\gamma - 1) mc^2 - \frac{k}{r} \quad -(1)$$

the force equation must also be considered. The force is defined by:

$$\int \underset{\sim}{F} \cdot d\underline{r} = (\gamma - 1) mc^2 \quad -(2)$$

where

$$\underline{F} = \frac{d}{dt}(\gamma m \underline{v}) \quad -(3)$$

and

$$\underline{p} = \gamma m \underline{v} \quad -(4)$$

is the relativistic momentum.

So:

$$\underline{F} = \gamma m \frac{d\underline{v}}{dt} + m \underline{v} \frac{d\gamma}{dt}, \quad -(5)$$

where

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv} \frac{dv}{dt} \quad -(6)$$

with

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(7)$$

So

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(8)$$

and

2) So

$$\underline{F} = \gamma m \frac{d\underline{v}}{dt} + m\gamma^3 \frac{v}{c^2} \underline{v} \quad -(9)$$

this is the rigorously correct relativistic force of the Sommerfeld theory. For:

$$v << c \quad -(10)$$

it is approximated by:

$$\underline{F} \sim \gamma m \frac{d\underline{v}}{dt} \quad -(11)$$

So in the Sommerfeld theory:

$$F = \gamma m (\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \quad -(12)$$

where

$$k = \frac{e^2}{4\pi\epsilon_0}. \quad -(13)$$

The velocity $v$ of the Sommerfeld theory is the same as that of the Bohr theory

$$v = \frac{\alpha_f}{nc} \quad -(14)$$

where $\alpha_f$ is the fine structure constant and $n$ is the main quantum number. The reason for this is that the Sommerfeld theory is based on the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - v^2 dt^2$$
$$= c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(15)$$

so v is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(16)$$

and

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(17)$$

It follows from eq. (12) that:

$$m\frac{d^2 r}{dt^2} = \frac{L^2}{mr^3} - \left(1 - \left(\frac{d_g}{nc}\right)^2\right)^{1/2} \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(18)$$

The Bohr velocity v is obtained from the fundamental assumptions of the Bohr theory of the atom:

$$\frac{L^2}{mr^3} = \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(19)$$

with

$$L = n\hbar \quad -(20)$$

so defining the Bohr radius:

$$r = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \quad -(21)$$

4) It follows from eqs. (18) and (19) that in the Sommerfeld theory:

$$\frac{dr}{dt} \neq 0 \quad -(22)$$

and the orbits are not circles. The velocity in the Sommerfeld theory is therefore:

$$v^2 = \left(\frac{\alpha g}{nc}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(23)$$

because

$$v^2(\text{Sommerfeld}) = v^2(\text{Bohr}) \quad -(24)$$

So in the Sommerfeld theory:

$$m\frac{d^2r}{dt^2} = \frac{e^2}{4\pi\epsilon_0 r^2}\left(1 - \left(1 - \left(\frac{\alpha g}{nc}\right)^2\right)^{1/2}\right)$$
$$\neq 0 \quad -(25)$$

In general, the orbit of the Sommerfeld theory is:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(26)$$

and is a precessing ellipse in general.

5) The force law of the Sommerfeld orbit is defined by:

$$F = \gamma m(\ddot{r} - r\dot{\theta}^2) = -\partial U/\partial r = -\frac{k}{r^2}$$

$$= -\frac{\gamma L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(27)$$

It follows that:

$$m\ddot{r} = -\frac{L^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(28)$$

and is the same format in the relativistic theory.

From eqs. (26) and (28):

$$m\ddot{r} = \frac{x^2 L^2}{mr^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right) \quad -(29)$$

Note that in a circle:

$$r = \alpha \quad -(30)$$

so in a circle: $m\ddot{r} = 0 \quad -(31)$

which is the result of Bohr theory with its circular orbits. In the Sommerfeld theory with its precessing elliptical orbits, eq. (29) is true.

From eq. (12):

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{1}{\gamma}\frac{k}{r^2} \quad -(32)$$

From eqs. (29) and (32):

$$x^2 = \left(\frac{L^2}{mr^3} - \frac{1}{\gamma}\frac{k}{r^2}\right)\left(\frac{L^2}{mr^3} - \frac{L^2}{\alpha mr^2}\right)^{-1} \quad -(33)$$

The precessing ellipse reduces to a static ellipse if

$$x = 1 \quad -(34)$$

in which case:

$$\frac{1}{\gamma}\frac{k}{r^2} = \frac{L^2}{\alpha mr^2} \quad -(35)$$

and

$$\alpha = \frac{\gamma L^2}{km} = \left(\frac{4\pi\epsilon_0 n^2\hbar^2}{me^2}\right)\gamma \quad -(36)$$

where

$$r_B = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2} \quad -(37)$$

is the Bohr radius.

So when:

$$\alpha = \gamma r_B \quad -(38)$$

then

$$x = 1 \quad -(39)$$

In this case eq. (29) reduces to:

$$m\ddot{r} = \frac{L^2}{mr^3} - \left(1 - \left(\frac{\alpha_g}{nc}\right)^2\right)^{1/2} \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(32)$$

$$= \frac{e^2}{4\pi\epsilon_0 r^2}\left(1 - \left(1 - \left(\frac{\alpha_g}{nc}\right)^2\right)^{1/2}\right)$$

which is eq. (25), QED, using the Bohr condition (19). In the Bohr theory:

$$\alpha = r_B \quad -(33)$$

and the ellipse reduces to a circle.

More generally, using the Bohr condition (19), eq. (33) reduces to:

$$\boxed{x c^2 = \left(1 - \frac{1}{\gamma}\right)\left(1 - \frac{r}{\alpha}\right)^{-1}} \quad -(34)$$

where

$$r = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \quad -(35)$$

and

$$\gamma = \left(1 - \left(\frac{\alpha_g}{nc}\right)^2\right)^{-1/2} \quad -(36)$$

## Summary of the Sommerfeld Theory

It is a quantization of the precessing ellipse of $x$ theory:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(37)$$

in which $x$ is defined by eq. (34). When

$$\alpha = \gamma r_B \quad -(38)$$

it reduces to:

$$r = \frac{\alpha}{1 + \epsilon \cos \theta} \quad -(39)$$

and when

$$\alpha = r_B \quad -(40)$$

it reduces to the circle

$$r = \alpha \quad -(41)$$

of Bohr theory, in which $\gamma$ is unity.

The velocity $v$ of the Sommerfeld and Bohr theories are the same:

$$\frac{v}{c} = \frac{\alpha_f}{n} \quad -(42)$$

and the Lorentz factor is defined by:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{dt}{d\tau} \quad -(43)$$

9) The total energy of the Sommerfeld theory is:

$$H = E = (\gamma - 1)mc^2 - \frac{k}{r} \quad - (44)$$

and its force law is:

$$F = \gamma m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \quad - (45)$$

if

$$v << c \quad - (46)$$

In the Sommerfeld theory:

$$\frac{d^2 r}{dt^2} \neq 0. \quad - (47)$$

<u>Summary of the Bohr Theory</u>

In general this is a quantization of the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad - (48)$$

in which

$$\alpha = \frac{L^2}{mk} \quad - (49)$$

and

$$\epsilon^2 = 1 + \frac{2E\alpha}{k} \quad - (50)$$

In this ellipse:

$$\alpha = r_B \quad - (51)$$

10) where $r_B$ is the Bohr radius. The Bohr radius is the half right latitude of the static ellipse. The Bohr theory of the atom is given by the limit of zero ellipticity:

$$\epsilon^2 = 0 = 1 + \frac{2E\alpha}{k} \quad -(52)$$

giving the energy levels of the Bohr atom:

$$E = -\frac{k}{2\alpha} = -\frac{k}{2r_B} = -\frac{me^4}{32\pi^2\epsilon_0^2 n^2 \hbar^2} \quad -(53)$$

These are the energy levels of circular Bohr orbits. The total energy of the Bohr theory is:

$$H = E = \frac{1}{2}mv^2 - \frac{k}{r} \quad -(54)$$

and the force law is:

$$F = m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \quad -(55)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(56)$$

and

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(57)$$