262(5): Spin Connection and Definition of Linear Velocity in Plane Polar Coordinates

In plane polar coordinates, linear velocity is defined by:

$$\underline{v} = \frac{dr}{dt}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(1)$$

$$= \frac{dr}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{r}$$

where

$$\underline{\omega} = \frac{d\theta}{dt}\,\underline{k} \quad -(2)$$

is the angular velocity. In eq. (1), dr/dt is the derivative with static axes. So eq. (1) means:

$$\frac{D\underline{r}}{dt} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad -(3)$$

The second term on the right hand side of eq. (3) means that the rotation of the axes of the plane polar coordinate system produces the orbital linear velocity $\underline{\omega} \times \underline{r}$. This cross product is defined as:

$$\underline{\omega} \times \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \quad -(4)$$

2)

$$= \underline{i}\left(\omega_2 r_3 - \omega_3 r_2\right) - \underline{j}\left(\omega_1 r_3 - \omega_3 r_1\right) + \underline{k}\left(\omega_1 r_2 - \omega_2 r_1\right) \quad -(5)$$

so the components of eq. (3) are:

$$D_\mu r^1 = \partial_\mu r^1 + \omega^2_\mu r^3 - \omega^3_\mu r^2 \quad -(6)$$

$$D_\mu r^2 = \partial_\mu r^2 + \omega^3_\mu r^1 - \omega^1_\mu r^3 \quad -(7)$$

$$D_\mu r^3 = \partial_\mu r^3 + \omega^1_\mu r^2 - \omega^2_\mu r^1 \quad -(8)$$

in which

$$\mu = 0 \quad -(9)$$

and

$$\underline{\omega} = \omega^1{}_0 \underline{i} + \omega^2{}_0 \underline{j} + \omega^3{}_0 \underline{k} \quad -(10)$$

$$\underline{r} = r^1 \underline{i} + r^2 \underline{j} + r^3 \underline{k} \quad -(11)$$

Consider eq. (6) and write:

$$\omega^1{}_{\mu 3} = \epsilon^{12}{}_3 \, \omega^2_\mu \quad -(12)$$

$$\omega^1{}_{\mu 2} = \epsilon^{13}{}_2 \, \omega^3_\mu \quad -(13)$$

where:

$$\epsilon^{12}{}_3 = -\epsilon^{13}{}_2 = 1 \quad -(14)$$

is the totally antisymmetric unit tensor in

3) three space dimensions. So eq. (6) becomes:

$$D_\mu r^1 = \partial_\mu r^1 + \omega^1_{\mu 3} r^3 + \omega^1_{\mu 2} r^2 \quad -(15)$$

which is an example of the Cartan covariant derivative:

$$D_\mu r^a = \partial_\mu r^a + \omega^a_{\mu b} r^b \quad -(16)$$

with: $a = 1, \; b = 3$ and 2. $\quad -(17)$

So eq. (1) is an example of eq. (16). and an example of Cartan's geometry, QED.

The Cartan spin connection in units of rad s^{-1} is eq. (10), i.e. is the angular velocity vector. In units of inverse metres the spin connection is ω / c.

Therefore eq. (1) of all the text books is an example of general relativity based on Cartan geometry, Q.E.D.