262(1): The Development of a Force Law for Any Orbit

Consider the orbital and spin force laws given in UFT 261:

$$\underset{\sim}{F}(\text{orb}) = -\underline{\nabla}\phi - \frac{\partial \underline{p}}{\partial t} - \omega_0 \underline{p} + \phi \underline{\omega} \quad -(1)$$

$$\underset{\sim}{F}(\text{spin}) = \underline{\nabla} \times \underline{p} - \underline{\omega} \times \underline{p} . \quad -(2)$$

By antisymmetry:

$$-\frac{\partial \underline{p}}{\partial t} - \omega_0 \underline{p} = -\underline{\nabla}\phi + \underline{\omega}\phi \quad -(3)$$

Eq. (3) is obtained from the definition of torsion in Cartan geometry:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(4)$$

where:

$$\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu \quad -(5)$$

$$\Gamma^a_{\nu\mu} = \partial_\nu q^a_\mu + \omega^a_{\nu b} q^b_\mu . \quad -(6)$$

By the commutator theorem of geometry:

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu} \quad -(7)$$

So:

2)
$$\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu = -\left(\partial_\nu q^a_\mu + \omega^a_{\nu b} q^b_\mu\right) \quad -(8)$$

For example:

$$\partial_0 q^a_1 + \omega^a_{0b} q^b_1 = -\left(\partial_1 q^a_0 + \omega^a_{1b} q^b_0\right) \quad -(9)$$

In vector notation, eq. (9) is:

$$-\frac{1}{c}\frac{\partial \underline{q}^a}{\partial t} - \omega^a_{0b}\underline{q}^b = -\underline{\nabla} q^a_0 + \underline{\omega}^a_b q^b_0 \quad -(10)$$

With the definitions:

$$\omega^a_{\mu b} = \left(\frac{1}{c}\omega^a_{0b}, -\underline{\omega}^a_b\right) \quad -(11)$$

and

$$\phi^a_0 = c q^a_0, \quad -(12)$$

$$p^a_\mu = \left(\frac{\phi^a_0}{c}, -\underline{p}^a\right) \quad -(13)$$

and using the one polarization model for simplicity only:

$$-\frac{\partial \underline{p}}{\partial t} - \omega_0 \underline{p} = -\underline{\nabla}\phi + \underline{\omega}\phi \quad -(14)$$

In the absence of a spin connection eq. (14) becomes:

$$-\frac{\partial \underline{p}}{\partial t} = -\underline{\nabla}\phi \quad -(15)$$

which is the equivalence of gravitational and inertial mass, tested experimentally to one part in ten power seventeen. Therefore eq. (14) is the generally covariant format of the equivalence theorem.

It is convenient to define the force as:

$$\underline{F} = -\frac{\partial \underline{p}}{\partial t} - \omega_0 \underline{p} = -\underline{\nabla}\phi + \underline{\omega}\phi \quad -(16)$$

and to compare this definition with the equivalent definition from the plane polar formulation of any planar orbit, in which the velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(17)$$

and the acceleration is:

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(18)$$

As argued in previous work eqs. (17) and (18) are examples of Cartan geometry because the axes of the plane polar system are <u>rotating</u>. So there is present a geometrical

4) connection and eqs. (17) and (18) are equations of general relativity for this reason. Eq. (16) is also an equation of general relativity. If the axes were not rotating then:

$$\underline{v} = \dot{r}\,\underline{e}_r \quad -(19)$$

$$\underline{a} = \ddot{r}\,\underline{e}_r \quad -(20)$$

and this is the Newtonian theory.

For any orbit in a plane, a lagrangian analysis shows that the angular momentum is a constant of motion and is conserved. It is given by:

$$L = m r^2 \dot{\theta} \quad -(21)$$

Therefore:

$$\dot{\theta} = \frac{L}{m r^2} \quad -(22)$$

and

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} \quad -(23)$$

Let

$$f(r) = \frac{L}{m r^2} \quad -(24)$$

then

$$\ddot{\theta} = \frac{df(r)}{dt} = \frac{df(r)}{dr}\frac{dr}{dt} \quad -(25)$$

So:

$$\ddot{\theta} = \dot{r}\frac{d}{dr}\left(\frac{L}{mr^2}\right) = -\frac{2L}{mr^3}\dot{r} \quad -(26)$$

So

$$r\ddot{\theta} = -\frac{2L}{mr^2}\dot{r} \quad -(27)$$

and

$$2\dot{r}\dot{\theta} = \frac{2L}{mr^2}\dot{r} \quad -(28)$$

It follows that the Coriolis force for any planar orbit vanishes:

$$\underline{F}_{Coriolis} = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$
$$= \underline{0} \quad -(29)$$

The force from eq. (18) for any planar orbit is therefore radial:

$$\underline{F} = m\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r$$
$$= \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m r^3}\right)\underline{e}_r \quad -(30)$$

The Lagrangian analysis produces

6) The equation:

$$F(r) = -\frac{L}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(31)$$

and this is again an equation of general relativity because it uses the rotating axes of the plane polar coordinate system

If the orbit is a static ellipse then:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos\theta\right) \quad -(32)$$

so

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\theta \quad -(33)$$

and

$$F(r) = -\frac{L}{mr^2\alpha} \quad -(34)$$

The orbit has no Coriolis force so:

$$\underset{\sim}{F}(r) = -\frac{L}{m\alpha r^2}\,\underline{e}_r \quad -(35)$$

From eqs (30) and (35):

$$\frac{d^2 r}{dt^2} - \frac{L^2}{mr^3} = -\frac{L}{m\alpha r^2} \quad -(36)$$

So:

$$\boxed{\frac{d^2 r}{dt^2} = -\frac{L}{m\alpha r^2} + \frac{L^2}{m r^3}} \quad -(37)$$

In a stable orbit:

$$\frac{d^2 r}{dt^2} = 0 \quad -(38)$$

so the inverse square law of attraction is balanced by the outward force proportional to $1/r^3$. This is the centrifugal force:

$$-\underline{\omega} \times (\underline{\omega} \times \underline{r}) m = \frac{L^2}{m r^3} \underline{e}_r \quad -(39)$$

$$= r \dot{\theta}^2 \underline{e}_r .$$

From vector analysis:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \underline{\omega} (\underline{\omega} \cdot \underline{r}) - \omega^2 \underline{r} . \quad -(40)$$

Since

$$\underline{\omega} = \frac{d\theta}{dt} \underline{k} \quad -(41)$$

$$-\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \omega^2 r \underline{e}_r \quad -(42)$$

which is eqn. (39), Q.E.D.

8) The centrifugal force is a result of the rotation of the axes in a plane polar coordinate system. It does not occur in Newtonian dynamics and is an example of Cartan geometry, which always governs physics. It is obviously a measurable force.

From eqs. (14) and (30):

$$\underline{F} = -\frac{d\underline{p}}{dt} - \omega \cdot \underline{p} = -\underline{\nabla}\phi + \underline{\omega}\phi$$

$$= \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m r^3} \right) \underline{e}_r \quad -(43)$$

and

$$\underline{F} = -\frac{L}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} \right) \underline{e}_r \quad -(44)$$

because there is no $\underline{e}_\theta$ dependent Coriolis force in any planar orbit, so $\underline{F}$ is always radial.

These equations are valid for any planar orbit including the orbit of a star in a whirlpool galaxy, and for all perihelion precessions.

The Newtonian result is:

$$\underline{F} = -\underline{\nabla}\phi = \frac{d^2 r}{dt^2}\,\underline{e}_r \quad -(45)$$

from eq. (43), and:

$$\underline{F} = -\frac{L}{m\alpha r^2}\,\underline{e}_r \quad -(46)$$

from eq. (44) for an elliptical orbit. So in Newtonian dynamics the centrifugal term is missing.

The centrifugal term is due to the spin connection, and preceding as in UFT261:

$$\phi\,\underline{\omega} = -\frac{L^2}{m r^3}\,\underline{e}_r \quad -(47)$$

In a near Newtonian approximation:

$$\phi = -\frac{mMG}{r} \quad -(48)$$

here m is attracted to M at the focus of an ellipse and G is the Newton constant.

So:

$$-\frac{mMG}{r}\underline{\omega} = -\frac{L^2}{mr^3}\underline{e}_r \quad -(49)$$

and the spin connection is:

$$\boxed{\underline{\omega} = \frac{L^2}{m^2 \underline{M} G r^2}\underline{e}_r} \quad -(50)$$

The right latitude α in the Newton theory is:

$$\alpha = \frac{L^2}{m^2 \underline{M} G} \quad -(51)$$

so

$$\boxed{\underline{\omega} = \frac{\alpha}{r^2}\underline{e}_r} \quad -(52)$$

Eq. (50) is true for all planar orbits, and eq. (52) for an ellipse.