

262(3) : Force Law for the Hyperbolic Spiral Orbit.

The force law may be obtained from a Lagrangian analysis where the Lagrangian is:

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r). \quad (1)$$

The Euler Lagrange equations are:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0 \quad (2)$$

and

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad (3)$$

Eq. (2) gives the conservation of angular momentum:

$$L = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{constant} \quad (4)$$

and eq. (3) gives the equivalence of inertial and gravitational mass:

$$m(\ddot{r} - r\dot{\theta}^2) = F(r) = -\frac{\partial U}{\partial r} \quad (5)$$

Note that the fundamental kinematic result is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2) \underline{e}_r \quad (6)$$

s. of kinematic and Lagrangian methods are self-consistent.

Now transform eq. (5) using calculus.

d) First use:

$$\frac{d}{d\theta} \left(\frac{1}{r} \right) = - \frac{1}{r^2} \frac{dr}{d\theta} = - \frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\theta} \quad - (7)$$

From eq. (4):

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad - (8)$$

so

$$\frac{d}{d\theta} \left(\frac{1}{r} \right) = - \frac{m}{L} \frac{dr}{dt} \quad - (9)$$

Now use:

$$\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) = \frac{d}{d\theta} \left(\frac{d}{d\theta} \left(\frac{1}{r} \right) \right) = - \frac{m}{L} \frac{d}{d\theta} \frac{dr}{dt} \quad - (10)$$

Let

$$f = \frac{d}{d\theta} \left(\frac{1}{r} \right) \quad - (11)$$

then

$$\frac{df}{d\theta} = \frac{df}{dt} \frac{dt}{d\theta} \quad - (12)$$

so

$$\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) = - \frac{m^2 r^2}{L^2} \frac{d^2 r}{dt^2} \quad - (13)$$

Therefore

$$\ddot{r} = \frac{d^2 r}{dt^2} = - \frac{L^2}{m^2 r^2} \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) \quad - (14)$$

and

$$r \dot{\theta}^2 = \frac{L^2}{m^2 r^3} \quad - (15)$$

Therefore eq. (5) can be transformed into:

$$F(r) = -\frac{L^2}{mr^3} \left(\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} \right) \quad - (16)$$

where

$$\underline{F}(r) = F(r) \underline{e}_r \quad - (17)$$

Eq. (16) can be used to find the force $F(r)$ for any orbit is a plane. Note carefully that it is more general than the Newton and Einstein theories, or any theory that asserts a force law a priori. Eq. (16) originates in the fundamental kinematics of plane polar coordinates. These kinematics are an example of Cartier geometry.

In note 262(2) it was shown that if the orbital linear velocity of any plane orbit becomes constant (V_∞) at infinite r then that limiting orbit must be the hyperbolic spiral:

$$\theta = -\left(\frac{L}{mV_\infty} \right) \frac{1}{r} \quad - (18)$$

this again is the result of the fundamental kinematics plane polar coordinates. In deriving eq. (18) force law was assumed. The force law can be calculated from eqs. (16) and (18). Use:

$$\frac{d^2}{dt^2} \left(\frac{1}{r} \right) = 0 \quad - (19)$$

for the hyperbolic spiral (18), so:

$$F(r) = -\frac{L^2}{mr^3} \quad - (20)$$

and

$$\frac{d^2 r}{dt^2} = \ddot{r} = 0 \quad - (21)$$

so

$$\frac{dv}{dt} = 0. \quad - (22)$$

The velocity is a constant as $r \rightarrow \infty$ as observed experimentally. The limiting orbit is the hyperbolic spiral (18) as observed experimentally. The main experimental features of a whirlpool galaxy can be explained by fundamental kinematics, i.e. by geometry.

The Newton and Einstein theories fail totally because they don't give:

$$v \xrightarrow{r \rightarrow \infty} 0. \quad - (23)$$

The gravitational potential is a whirlpool

3) galaxy is given by Eq. (5) wth eq. (21):

$$-\frac{dU(r)}{dr} = -\frac{L^2}{mr^3} \quad - (24)$$

So

$$U(r) = \int \frac{L^2}{mr^3} dr \\ = -\frac{L^2}{2mr^2} \quad - (25)$$

and this is different from the Newtonian potential:

$$U(r) (\text{Newton}) = -\frac{mM_G}{r} \quad - (26)$$

The Newton theory gives:

$$\frac{d^2}{dt^2} \left(\frac{1}{r} \right) + \frac{1}{r} = \frac{m^2 M_G}{L^2} \quad - (27)$$

(Maria and Thoma eq. (7.73)) and the
Einstein theory gives:

$$\frac{d^2}{dt^2} \left(\frac{1}{r} \right) + \frac{1}{r} = \frac{m^2 M_G}{L^2} + \frac{3GM}{c^2 r^2} \quad - (28)$$

(Maria and Thoma eq. (7.74)).

The Newton and Einstein theories contain M

i) but the correct result (25) for a whirlpool galaxy
does not contain M .

This means that the velocity of a star
in a whirlpool galaxy reads a constant value
irrespective of the mass M at the centre of the
galaxy. This is completely not Newtonian
and completely not-Einsteinian, but is
explained straightforwardly by fundamental
geometry, the kinematics of the plane polar coordinates.

This geometry is of course an example of
Einstein's general geometry.

Finally Newton gives the orbit:

$$r(\text{Newton}) = \frac{\alpha}{1 + \epsilon \cos \theta} \quad - (29)$$

and Einstein gives the orbit:

$$r(\text{Einstein}) = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad - (30)$$

where

$$x \approx 1. \quad - (31)$$

Both orbits fail totally in a whirlpool galaxy
