Note 261(5), Diagrams of Hodge Duality

The most well known Hodge duality is:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} \rightarrow \tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{bmatrix}$$

Example, the E^1 element

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ E^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E^1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 01 \rightarrow 23$$

It has moved from 01 to 23, it has changed position within the same matrix. In tensor notation this means that:

$$\epsilon^{0123} = 1 \quad -(1)$$

where ϵ^{0123} is the totally antisymmetric unit tensor in four dimensions. This is an abstract and difficult concept but it has a simple effect.

Similarly:

2)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & E^2 & 0 & 0 \end{bmatrix}, \quad 02 \rightarrow 31$$

$$\epsilon^{0231} = 1 \quad -(2)$$

So element 02 has moved to element 31 with the same matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E^3 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & E^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 03 \rightarrow 12$$

$$\epsilon^{0312} = 1 \quad -(3)$$

So element 03 has moved to 12 with the same matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & B^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B^3 & 0 & 0 & 0 \end{bmatrix}, \quad 21 \rightarrow 03$$

So element 21 has moved to 03 with the same matrix.

The overall result is exemplified by:

$$F^{10} \rightarrow \tilde{F}^{32} \quad -(4)$$
$$F^{20} \rightarrow \tilde{F}^{13} \quad -(5)$$
$$F^{30} \rightarrow \tilde{F}^{21} \quad -(6)$$

So:

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \partial_1 \tilde{F}^{32} + \partial_2 \tilde{F}^{13} + \partial_3 \tilde{F}^{21}$$
$$= \partial_1 \tilde{F}_{32} + \partial_2 \tilde{F}_{13} + \partial_3 \tilde{F}_{21} \quad -(7)$$

In tensor notation eq. (7) is an example of

$$d \wedge \tilde{F} = \partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} \quad -(8)$$

$$F^{32} \rightarrow \tilde{F}^{10} \quad -(7)$$
$$F^{13} \rightarrow \tilde{F}^{20} \quad -(8)$$
$$F^{21} \rightarrow \tilde{F}^{30} \quad -(9)$$

So

$$\partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30}$$
$$= \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} \quad -(10)$$
$$= \partial_1 F_{32} + \partial_2 F_{13} + \partial_3 F_{21}$$

In tensor notation eq. (10) is an example of:

4)

$$d \wedge F = \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \quad -(11)$$

Formally, these equations are derived from:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad -(12)$$

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \quad -(13)$$

in the Minkowski spacetime:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(14)$$

It is seen that:

$$d \wedge F = \partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \quad -(15)$$

and

$$d \wedge \tilde{F} = \partial_\mu F^{\mu\nu} = \partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} \quad -(16)$$

$\boxed{\text{The existence of } d \wedge F \text{ implies the existence of } d \wedge \tilde{F}\text{, QED.}}$

In the theory of Maxwell Heaviside

electromagnetism this is very well known. In free space:

$$d \wedge \tilde{F} = 0 \quad -(17)$$

$$d \wedge F = 0 \quad -(18)$$

In field matter interaction:

$$d \wedge \tilde{F} = J \quad -(19)$$

$$d \wedge F = 0 \quad -(20)$$

Eq. (18) is an example of the Cartan identity and Eq. (17) is an example of the Evans identity in ECE theory. They are examples with zero spin connection and zero curvature.

The complete Cartan identity is:

$$D \wedge T = d \wedge T + \omega \wedge T := q \wedge R \quad -(21)$$

where, for each a:

$$d \wedge T = \partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T_{\nu\rho} + \partial_\rho T_{\mu\nu} + \partial_\nu T_{\rho\mu} \quad -(22)$$

$$d \wedge \tilde{T} = \partial_\mu T^{\mu\nu} = \partial_\mu \tilde{T}_{\nu\rho} + \partial_\rho \tilde{T}_{\mu\nu} + \partial_\nu \tilde{T}_{\rho\mu} \quad -(23)$$

The wedge product of ω and T is:

$$\omega \wedge T = \omega^a_{\mu b} T^b_{\nu\rho} + \omega^a_{\rho b} T^b_{\mu\nu} + \omega^a_{\nu b} T^b_{\rho\mu}$$

$$= \omega^a_{\mu b} \tilde{T}^{b\mu\nu} \quad -(24)$$

In precise analogy with the arguments leading to eqns. (22) and (23) it follows that:

$$\omega \wedge \tilde{T} = \omega^a_{\mu b} \tilde{T}^b_{\nu\rho} + \omega^a_{\rho b} \tilde{T}^b_{\mu\nu} + \omega^a_{\nu b} \tilde{T}^b_{\rho\mu}$$

$$= \omega^a_{\mu b} T^{b\mu\nu} \quad -(25)$$

Finally the wedge product of q and R is the wedge product of a one form and two form:

$$q \wedge R = q^b_\mu R^a_{b\nu\rho} + q^b_\rho R^a_{b\mu\nu} + q^b_\nu R^a_{b\rho\mu}$$

$$= q^b_\mu \tilde{R}^{a\ \mu\nu}_{\ b} \quad -(26)$$

$$= \tilde{R}^{a\ \mu\nu}_{\ \mu}$$

It follows that:

$$q \wedge \tilde{R} = R^{a\ \mu\nu}_{\ \mu} \quad -(27)$$

$$= q^b_\mu \tilde{R}^a_{b\nu\rho} + q^b_\rho \tilde{R}^a_{b\mu\nu} + q^b_\nu \tilde{R}^a_{b\rho\mu}$$

7)

The overall result is that eq. (21) implies:

$$d \wedge \tilde{T} + \omega \wedge \tilde{T} := q \wedge \tilde{R} \quad -(28)$$

The Hodge duality in this case is:

$$\tilde{T}^{a\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} T^{a}{}_{\alpha\beta} \quad -(29)$$

$$\tilde{R}^{a}{}_{b}{}^{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} R^{a}{}_{b\alpha\beta} \quad -(30)$$

where:

$$g = |g_{\mu\nu}| \quad -(31)$$

Eq. (28) is the Evans identity:

$$D \wedge \tilde{T} := q \wedge \tilde{R} \quad -(32)$$
