261(7): Antisymmetry Law applied to Momentum, Equivalence Principle, and Light Deflection due to Gravitation

The torsion is defined by:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(1)$$

with antisymmetry:

$$\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu = -\left(\partial_\nu q^a_\mu + \omega^a_{\nu b} q^b_\mu\right) \quad -(2)$$

i.e

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu} \quad -(3)$$

The orbital part of eq. (1) is:

$$\partial_0 q^a_1 + \omega^a_{0b} q^b_1 = -\left(\partial_1 q^a_0 + \omega^a_{1b} q^b_0\right) \quad -(4)$$

When this is applied to momentum:

$$\partial_0 p^a_1 + \omega^a_{0b} p^b_1 = -\left(\partial_1 p^a_0 + \omega^a_{1b} p^b_0\right) \quad -(5)$$

where:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right), \quad -(6)$$

$$p_\mu = \left(p_0, -\underline{p}\right). \quad -(7)$$

Therefore:

2)

$$p^a_1 = -p^a_x \quad -(8)$$

and:

$$-\frac{\partial p^a}{\partial t} - c\omega^a{}_{0b}\,\underline{p}^b = -c\underline{\nabla}\,p^a_0 + c\underline{\omega}^a{}_b\,p^b_0$$
$$-(9)$$

In analogy with the electromagnetic potential:

$$A^a_\mu = \left(A^a_0, -\underline{A}^a\right) \quad -(10)$$
$$= \left(\frac{\phi^a}{c}, -\underline{A}^a\right)$$

the momentum tetrad is defined by:

$$p^a_\mu = \left(p^a_0, -\underline{p}^a\right) \quad -(11)$$
$$= \left(\frac{\Phi^a_0}{c}, -\underline{p}^a\right)$$

from the minimal prescription:

$$p^a_\mu \rightarrow p^a_\mu + eA^a_\mu \quad -(12)$$

so

$$\Phi^a = e\phi^a \quad -(13)$$
$$\underline{p}^a = e\underline{A}^a \quad -(14)$$

Therefore
$$-\frac{\partial p^a}{\partial t} - c\omega^a{}_{0b}\,\underline{p}^b = -\underline{\nabla}\,\Phi^a + \underline{\omega}^a{}_b\,\Phi^b$$
$$-(15)$$

). Define the force as:

$$\underline{F}^a = -\frac{d\underline{p}^a}{dt} - c\omega^a{}_{0b}\,\underline{p}^b = -\underline{\nabla}\,\underline{\Phi}^a + \underline{\omega}^a{}_b\,\underline{\Phi}^b \quad -(16)$$

so $\underline{\Phi}^a_0$ is the gravitational potential.

In the absence of the spin connection:

$$\underline{F}^a = -\frac{d\underline{p}^a}{dt} = -\underline{\nabla}\,\underline{\Phi}^a \quad -(17)$$

and for each a:

$$\underline{F} = -\frac{d\underline{p}}{dt} = -\underline{\nabla}\,\underline{\Phi} \quad -(18)$$

which is the equivalence principle. In Newtonian physics:

$$\underline{\Phi} = -mMG/r, \quad -(19)$$

$$\underline{p} = m\underline{v}, \quad -(20)$$

so

$$\underline{F} = -m\underline{g} = -\underline{\nabla}\,\underline{\Phi} \quad -(21)$$

$$= -(mMG/r^2)\,\underline{e}_r$$

so

$$g = \frac{MG}{r^2} \quad -(22)$$

is the acceleration due to gravity.

4) In order to calculate the light deflection due to gravity use the experimental fact that all planar orbits are represented by a precessing conical section:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(23)$$

for small x. As in UFT 215 and UFT 216 Eq. (23) can be used in the equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L} F(r) \quad -(24)$$

where L is the conserved total angular momentum (Marion and Thornton eq. (7.21)). Eq. (24) is more general than Newton and is derived from Lagrangian dynamics. It is valid for any orbit.

From eqs. (23) and (24):

$$F(r) = -\frac{k x^2}{r^2} - \frac{k(1 - x^2)\alpha}{r^3} \quad -(25)$$

where k is a constant.

If:

$$\underline{p} = p_r \, \underline{e}_r \quad -(26)$$

$$\underline{\omega} = \omega_r \, \underline{e}_r \quad -(27)$$

then:

$$F = -\frac{\partial \Phi}{\partial r} + \omega_r \Phi \quad -(28)$$

$$= -\frac{k x^2}{r^2} - \frac{k(1 - x^2)\alpha}{r^3}$$

5) For small deviation from a Newtonian orbit:

$$-\frac{\partial \Phi}{\partial r} = -\frac{k x^2}{r^2} \quad -(29)$$

and

$$x \sim 1 \quad -(30)$$

so:

$$\Phi \, \omega_r = \frac{k(x^2-1)\alpha}{r^3} \quad -(31)$$

to an excellent approximation, where:

$$\Phi = -\frac{k}{r}, \quad -(32)$$

$$k = mMG. \quad -(33)$$

So for the precession of the perihelion:

$$\boxed{\omega_r = -(1-x^2)\frac{\alpha}{r^2}} \quad -(34)$$

where α is the half right latitude of the orbit:

$$\alpha = b^2 / a \quad -(35)$$

where a and b are the major and minor semi axes.

In light deflection by the sun the orbit is a hyperbola whose total deflection as in UFT 216 is:

6)

$$\Delta\phi = 2\sin^{-1}\frac{1}{\epsilon} \quad -(36)$$

As shown in UFT 216 for small angles of deflection at closest approach R_0:

$$\sin\phi \sim \phi = \frac{1}{\epsilon} = \left[\frac{m^2\alpha R_0}{x^2L^2}\left(v^2 - \frac{L^2}{m^2}\left(\frac{x^2-1}{R_0^2}\right)\right) - 1\right]^{-1} \quad -(37)$$

$$= \left[\frac{m^2\alpha R_0 v^2}{x^2L^2} - \frac{\alpha}{R_0}\left(\frac{x^2-1}{x^2}\right) - 1\right]^{-1}$$

The deflection is only microradians so:

$$2\phi = 2\left(\frac{m^2\alpha R_0 v^2}{x^2L^2} + \frac{\alpha}{R_0}\left(\frac{1-x^2}{x^2}\right)\right)$$

to an excellent approximation. $-(38)$

The Newtonian result is given by:

$$x = 1, \quad -(39)$$

$$\frac{m^2\alpha}{L^2} = \frac{1}{M} \quad -(40)$$

and for a photon:

$$v \to c \quad -(41)$$

so

$$2\phi = \frac{2MG}{R_0c^2} \quad -(42)$$

7) By experimental result, the value of light deflection by any mass $\underline{M}$ is:

$$2\phi = \frac{4MG}{R_0 c^2} \quad -(43)$$

and is twice the Newtonian value. Therefore:

$$\frac{2MG}{x R_0 c^2} + \frac{\alpha}{R_0}\left(\frac{1-x^2}{x^2}\right) = \frac{4MG}{R_0 c^2} \quad -(44)$$

so

$$\frac{\alpha}{R_0}\left(\frac{1-x^2}{x^2}\right) = \frac{4MG}{R_0 c^2} - \frac{2MG}{R_0 c^2 x}$$

$$= \frac{2MG}{R_0 c^2}\left(2 - \frac{1}{x}\right) \quad -(45).$$

Now use

$$\frac{\alpha}{R_0} = 1 + \epsilon \quad -(46)$$

so

$$\frac{1-x^2}{x^2} = \frac{2MG}{R_0 c^2 (1+\epsilon)}\left(\frac{2x-1}{x}\right) \quad -(47)$$

which gives the quadratic:

$$x^2(1+2a) - ax - 1 = 0 \quad -(48)$$

where

$$a = \frac{2MG}{R_0 c^2 (1+\epsilon)} \quad -(49)$$

8) The solution of the quadratic is:

$$x = \frac{1}{2(1+2a)}\left(a \pm \left(a^2 + 4(1+2a)\right)^{1/2}\right) \quad -(50)$$

However:

$$a << 1 \quad -(51)$$

because ϵ is very large and $2MG/(R_0 c^2)$ is of the order of microradians. So to an excellent approximation:

$$x \sim 1, \quad -(52)$$

and

$$2\phi = \frac{2R_0c^2}{MG} + 2(1+\epsilon)\left(\frac{1-x^2}{x^2}\right). \quad -(53)$$

Experimentally:

$$2(1+\epsilon)\left(\frac{1-x^2}{x^2}\right) = \frac{2R_0c^2}{MG} \quad -(54)$$

where

$$\frac{1}{\epsilon} = \sin\left(\frac{\Delta\phi}{2}\right). \quad -(55)$$

For small deflections:

$$\frac{1}{\epsilon} = \frac{\Delta\phi}{2} \quad -(56)$$

so

$$\left(1 + \frac{2}{\Delta\phi}\right)\left(\frac{1-x^2}{x^2}\right) = \frac{R_0c^2}{MG} \quad -(57)$$

9) From eqns. (52) and (57):

$$1 - x^2 = \frac{R_0 c^2}{MG}\left(1 + \frac{2}{\Delta\phi}\right)^{-1} \quad - (58)$$

However:

$$\Delta\phi = \frac{4 R_0 c^2}{MG} \quad - (59)$$

so

$$1 - x^2 = \frac{\Delta\phi}{4}\left(1 + \frac{2}{\Delta\phi}\right)^{-1} \quad - (60)$$

The spin connection is therefore:

$$\omega_r = -\frac{\Delta\phi}{4}\left(1 + \frac{2}{\Delta\phi}\right)^{-1}\frac{\alpha}{r^2}, \quad - (61)$$

where:

$$\alpha = R_0(1+\epsilon) = R_0\left(1 + \frac{2}{\Delta\phi}\right) \quad - (62)$$

so

$$\boxed{\omega_r = -\frac{\Delta\phi}{4}\frac{R_0}{r^2}} \quad - (63)$$

At distance of closest approach:

$$r = R_0 \quad - (64)$$

so

$$\omega_r = -\frac{\Delta\phi}{4 R_0} \quad - (65)$$