Note 261(4): Development of Dynamics with the Vector Formulation of Cartan Geometry.

In vector formulation the first Maurer Cartan structure equation becomes two equations, respectively for orbital and spin torsion:

$$\underline{T}^a(\text{orb}) = -\underline{\nabla} q^a_{\ 0} - \frac{d\underline{q}^a}{dt} - \omega^a_{\ 0b}\underline{q}^b + q^b_{\ 0}\underline{\omega}^a_{\ b} \quad -(1)$$

and

$$\underset{\sim}{T}^a(\text{spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a_{\ b} \times \underline{q}^b \quad -(2)$$

The space part of the Cartan identity becomes the vector identity:

$$\underline{\nabla} \cdot \underline{\omega}^a_{\ b} \times \underline{q}^b = \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a_{\ b} - \underline{\omega}^a_{\ b} \cdot \underline{\nabla} \times \underline{q}^b$$

$$= -\underline{\nabla} \cdot \underset{\sim}{T}^a(\text{spin}) \quad -(3)$$

ecause:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{q}^a = 0. \quad -(4)$$

So:

$$\underline{\nabla} \cdot \underset{\sim}{T}^a(\text{spin}) = \underline{\omega}^a_{\ b} \cdot \underline{\nabla} \times \underline{q}^b - \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a_{\ b} \quad -(5)$$

In the special case:

$$\underline{\nabla} \cdot \underset{\sim}{T}^a(\text{spin}) = 0 \quad -(6)$$

then the Cartan identity simplifies to:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = 0 \quad -(7)$$

which has the Beltrami structure:

$$\underline{\nabla} \times \underline{\omega}^a{}_b \times \underline{q}^b = \kappa\, \underline{\omega}^a{}_b \times \underline{q}^b. \quad -(8)$$

Eq. (6) has the Beltrami structure:

$$\underline{\nabla} \times \underset{\sim}{T}^a(\text{spin}) = \kappa\, \underset{\sim}{T}^a(\text{spin}) \quad -(9)$$

It follows that:

$$\begin{aligned}\underline{\nabla} \times \underset{\sim}{T}^a(\text{spin}) &= \kappa\, \underset{\sim}{T}^a(\text{spin}) \\ &= \kappa\left(\underline{\nabla} \times \underline{q}^a - \underset{\sim}{\omega}^a{}_b \times \underline{q}^b\right) \\ &= \underline{\nabla} \times \underline{\nabla} \times \underline{q}^a - \underline{\nabla} \times \underline{\omega}^a{}_b \times \underline{q}^b\end{aligned} \quad -(10)$$

From eqs. (8) and (10):

$$\underline{\nabla} \times \underline{\nabla} \times \underline{q}^a = \kappa\, \underline{\nabla} \times \underline{q}^a \quad -(11)$$

So

$$\underline{\nabla} \times \underline{q}^a = \kappa\, \underline{q}^a \quad -(12)$$

5) In the special case:

$$\underline{\nabla} \cdot \underline{q}^a = 0 \quad -(13)$$

eqs. (12) and (13) give the Helmholtz equation

$$(\nabla^2 + \kappa^2)\, \underline{q}^a = \underline{0} \quad -(14)$$

The tetrad postulate is:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0$$

$$-(15)$$

where

$$\omega^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu \quad -(16)$$

$$\Gamma^a_{\mu\nu} = \Gamma^\lambda_{\mu\nu} q^a_\lambda. \quad -(17)$$

So

$$\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(18)$$

and

$$\Box q^a_\nu = \partial^\mu \partial_\mu q^a_\nu = \partial^\mu (\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu})$$

$$:= -R q^a_\nu \quad -(19)$$

where

$$R := q^\nu_a \partial^\mu (\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}) \quad -(20)$$

So the tetrad postulate is:

$$(\Box + R)\, q^a_\nu = 0 \quad -(21)$$

where:

$$\underset{\sim}{q}^a = \left(q_0^a , -\underline{q}^a \right) \quad -(22)$$

and

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \quad -(23)$$

It follows that:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + R \right) \underline{q}^a = \underline{0} \quad -(24)$$

Eq. (24) is a very fundamental result of geometry whereas eq. (14) is a special case. If it is assumed that:

$$\underline{q}^a = \underline{q}^a(0)\, e^{i\omega t} \quad -(25)$$

then:

$$\frac{1}{c^2}\frac{\partial^2 \underline{q}^a}{\partial t^2} = -\frac{\omega^2}{c^2}\underline{q}^a \quad -(26)$$

so

$$\left(\nabla^2 + \frac{\omega^2}{c^2} - R \right) \underline{q}^a = 0 \quad -(27)$$

Comparing eqs (14) and (27):

$$\kappa^2 = \frac{\omega^2}{c^2} - R \quad -(28)$$

or:

$$\omega^2 = c^2\kappa^2 + c^2 R \quad -(29)$$

This becomes the Einstein energy equation:

$$E^2 = \gamma mc^2$$
$$= c^2p^2 + m^2c^4 \quad -(30)$$

if

$$E = \hbar\omega, \; p = \hbar\kappa, \quad -(31)$$

and

$$R = \left(\frac{mc}{\hbar}\right)^2. \quad -(32)$$

The transformation from special to general relativity is therefore:

$$\left(\frac{mc}{\hbar}\right)^2 \rightarrow q_a^{\nu} \partial^{\mu}\left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) \quad -(33)$$

Now define the linear momentum tetrad by:

$$p^a_{\mu} = p^{(0)} q^a_{\mu} \quad -(34)$$

This equation follows from the ECE postulate

6)

$$A_\mu^a = A^{(0)} q_\mu^a \quad -(35)$$

using the minimal prescription:

$$p_\mu^a \rightarrow p_\mu^a + e A_\mu^a \quad -(36)$$

It follows that the orbital force is:

$$\underset{\sim}{F}^a(\text{orb}) = -\underline{\nabla}\phi_0^a - \frac{\partial \underset{\sim}{p}^a}{\partial t} - \omega^a{}_{0b}\underline{p}^b + \phi_0^b\underline{\omega}^a{}_b \quad -(37)$$

and the spin force is:

$$\underset{\sim}{F}^a(\text{spin}) = \underline{\nabla} \times \underset{\sim}{p}^a - \underline{\omega}^a{}_b \times \underline{p}^b \quad -(38)$$

In the single polarization theory:

$$\underset{\sim}{F}(\text{orb}) = -\underline{\nabla}\phi - \frac{\partial \underline{p}}{\partial t} - \omega_0 \underline{p} + \phi\underline{\omega} \quad -(39)$$

and

$$\underset{\sim}{F}(\text{spin}) = \underline{\nabla} \times \underline{p} - \underline{\omega} \times \underline{p} \quad -(40)$$

The Newtonian dynamics are recovered

from these equations in the limit of vanishing spin connection:

$$\underset{\sim}{F}(\text{orb}) = -\underline{\nabla}\phi - \frac{\partial \underline{p}}{\partial t} \quad -(40)$$

$$\underset{\sim}{F}(\text{spin}) = \underline{\nabla} \times \underline{p} \quad -(41)$$

By the antisymmetry law of ECE theory:

$$-\underline{\nabla}\phi = \frac{\partial \underline{p}}{\partial t} \quad -(42)$$

and the equivalence principle is derived from Cartan geometry:

$$\underset{\sim}{F} = \frac{\partial \underline{p}}{dt} = -m\underline{\nabla}\phi \quad -(43)$$

For Newtonian dynamics:

$$\phi = -\frac{GM}{r} \quad -(44)$$

$$\underline{\nabla}\phi = \frac{GM}{r^2} \quad -(45)$$

and

$$\underline{p} = m\underline{a} \quad -(46)$$

8) ECE theory has proven from geometry that the gravitational and inertial mass is the same:

$$\underset{\sim}{F} = m\underset{\sim}{a} = -mMG\frac{\underline{r}}{r^3} \quad -(47)$$

This is the first time that this fundamental theorem has been proven.

In the special cases:

$$\underline{\nabla} \cdot \underset{\sim}{F}(\text{spin}) = 0 \quad -(48)$$

and

$$\underline{\nabla} \cdot \underline{p} = 0 \quad -(49)$$

it follows that:

$$(\nabla^2 + \kappa^2)\underline{p} = 0 \quad -(50)$$

and

$$(\Box + R)p^a_\mu = 0 \quad -(51)$$

Eq. (50) leads to the free particle Schroedinger equation. As shown in UFT 260 the general Schroedinger equation is recovered in the more general case:

$$\underline{\nabla} \cdot \underline{p} \neq 0 \quad -(52)$$

The spin force is related to the vorticity $\underline{\nabla} \times \underline{v}$:

$$F(\text{spin}) = m\,\underline{\nabla} \times \underline{v} \quad -(53)$$