1)

Note 261(2): Example of Hodge Duals

In the Maxwell Heaviside (MH) theory:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad -(1)$$

so, for example:

$$\tilde{F}^{01} = F_{23} \quad -(2)$$

because

$$\epsilon^{0123} = 1. \quad -(3)$$

From eq. (2) it is entirely clear that $\tilde{F}^{01}$ and F_{23} are both elements of an antisymmetric 4x4 tensor in four dimensional space.

In free space:

$$d \wedge F = 0 \quad -(4)$$

$$d \wedge \tilde{F} = 0 \quad -(5)$$

Eq. (4) is a Cartan identity without curvature and without spin connection. Eq. (5) is an Evans identity of the same type.

Both F and $\tilde{F}$ are scalar valued two-forms of differential geometry. They are interchangeable because both are two-forms. So it follows that:

2)

$$d \wedge F = 0$$
$$\downarrow$$
$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$$
$$\downarrow$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$
$$\downarrow$$
$$\underline{\nabla} \cdot \underline{B} = 0$$
$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0}$$

$$d \wedge \tilde{F} = 0$$
$$\downarrow$$
$$\partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} = 0$$
$$\downarrow$$
$$\partial_\mu F^{\mu\nu} = 0$$
$$\downarrow$$
$$\underline{\nabla} \cdot \underline{E} = 0$$
$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0}$$

In ECE theory the Hodge dual transform takes place in the general space. In four dimensions it is:

$$\tilde{F}^{a\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} F^a_{\alpha\beta} \quad -(6)$$

where $\|g\|$ is the positive part or modulus of the determinant of the metric. In Minkowski spacetime:

$$\|g\| = 1 \quad -(7)$$

but this is no longer true in the general space.

So for example:

$$\tilde{F}^{a01} = \|g\| F^{a}_{23} \quad -(8)$$

The factor $\|g\|$ must be present to ensure that $\epsilon^{\mu\nu\alpha\beta}$ is a tensor, otherwise it is a symbol. Since $\|g\|$ is always a positive valued scalar it follows that $\tilde{F}^{a01}$ and F^{a}_{23} are both elements of a vector valued two-form, vector valued because of the presence of a. Both are elements of a 4×4 anti-symmetric tensor in four dimensions.

Since $\tilde{T}^{a}_{\mu\nu}$ is a vector valued two-form, the Evans identity follows immediately, and is an example of the Cartan identity.

The Cartan identity is:

$$D \wedge T^{a} := R^{a}_{b} \wedge q^{b} \quad -(9)$$

$$D_\mu T^{a}_{\nu\rho} + D_\rho T^{a}_{\mu\nu} + D_\nu T^{a}_{\rho\mu}$$

$$:= R^{a}_{\mu\nu\rho} + R^{a}_{\rho\mu\nu} + R^{a}_{\nu\rho\mu} \quad -(10)$$

4)

i.e.

$$D_\mu \left(\|g\|^{1/2} \tilde{T}^{a\mu\nu} \right) = \|g\|^{1/2} \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} \quad -(11)$$

By metric compatibility:

$$D_\mu \|g\|^{1/2} = 0 \quad -(12)$$

because metric compatibility is true for every element of the metric:

$$D_\mu g_{\nu\rho} = 0 \quad -(13)$$

and therefore is true for the determinant of the metric. So:

$$\|g\|^{1/2} D_\mu \tilde{T}^{a\mu\nu} = \|g\|^{1/2} \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} \quad -(14)$$

so

$$\boxed{D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a}{}_{\mu}{}^{\mu\nu}} \quad -(15)$$

This gives the homogeneous field equation of ECE theory. For electrodynamics:

$$D_\mu \tilde{F}^{a\mu\nu} = A^{(0)} \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} \quad -(16)$$

5) Similarly:

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b$$

$$\downarrow \qquad -(17)$$

$$D_\mu T^{a\mu\nu} := R^a{}_\mu{}^{\mu\nu}$$

$$\downarrow$$

$$D_\mu F^{a\mu\nu} := A^{(0)} R^a{}_\mu{}^{\mu\nu} \qquad -(18)$$

giving the inhomogeneous equations of ECE theory.
