260(7): Development of the Subsidiary Condition

As shown in note 260(6) the subsidiary condition is

$$\underline{p} \cdot \underline{\nabla} (\underline{\nabla} \cdot \underline{p}) = 0 \quad -(1)$$

Under this condition the Beltrami momentum equation

$$\underline{\nabla} \times \underline{p} = \kappa \underline{p} \quad -(2)$$

gives the Schroedinger equation:

$$(\nabla^2 + \kappa^2) \phi = 0 \quad -(3)$$

where

$$\kappa^2 = \frac{2m}{\hbar^2} (V - E) \quad -(4)$$

Two possible solutions of eq. (1) are:

$$\underline{\nabla} \cdot \underline{p} = 0 \quad -(5)$$

and

$$\underline{\nabla} (\underline{\nabla} \cdot \underline{p}) = 0 \quad -(6)$$

with

$$\underline{p} = -i\hbar \underline{\nabla} \quad -(7)$$

Eq. (5) gives:

$$\nabla^2 \phi = 0 \quad -(8)$$

which is consistent with eq. (3) only if

$$\kappa = 0 \quad -(8)$$

2) Eq. (6) give:

$$\underline{\nabla}\left(\nabla^2\phi\right) = 0 \quad -(9)$$

where

$$\nabla^2\phi = -\kappa^2\phi \quad -(10)$$

so

$$\underline{\nabla}\left(\kappa^2\phi\right) = 0 \quad -(11)$$

$$= \left(\underline{\nabla}\kappa^2\right)\phi + \kappa^2\underline{\nabla}\phi .$$

Therefore:

$$\boxed{\underline{\nabla}\phi = -\left(\frac{\underline{\nabla}\kappa^2}{\kappa^2}\right)\phi} \quad -(12)$$

and

$$\underline{\nabla}\cdot\underline{\nabla}\phi = \nabla^2\phi = -\underline{\nabla}\cdot\left(\frac{\underline{\nabla}\kappa^2}{\kappa^2}\phi\right)$$

$$= -\left(\left(\underline{\nabla}\cdot\left(\frac{\underline{\nabla}\kappa^2}{\kappa^2}\right)\right)\phi + \left(\frac{\underline{\nabla}\kappa^2}{\kappa^2}\right)\cdot\underline{\nabla}\phi\right) \quad -(13)$$

$$= -\left(\underline{\nabla}\cdot\frac{\underline{\nabla}\kappa^2}{\kappa^2} + \frac{\underline{\nabla}\kappa^2}{\kappa^2}\cdot\frac{\underline{\nabla}\kappa^2}{\kappa^2}\right)\phi$$

From a comparison of eqns. (3) and (13) we obtain the subsidiary condition:

$$\kappa^2 = \nabla \cdot \frac{\nabla \kappa^2}{\kappa^2} - \frac{\nabla \kappa^2 \cdot \nabla \kappa^2}{\kappa^4} \quad -(14)$$

$$= \frac{\nabla^2 \kappa^2}{\kappa^2} - \frac{\nabla \kappa^2 \cdot \nabla \kappa^2}{\kappa^4}$$

i.e.

$$\boxed{\kappa^2 \nabla^2 \kappa^2 = \nabla \kappa^2 \cdot \nabla \kappa^2 + \kappa^6} \quad -(15)$$

where

$$\kappa^2 = \frac{2m}{\hbar^2}\left(V - E\right) \quad -(16)$$

Eqs. (15) and (16) define the potential V needed for the momentum Beltrami equation to become a Schroedinger equation.

Finally, this potential is used to calculate the scattering amplitude in the Born approximation:

$$f(\hat{\boldsymbol{\kappa}}') = -\frac{m}{2\pi\hbar^2}\int e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}'}\, V(\mathbf{r}')\, e^{i\boldsymbol{\kappa}\cdot\mathbf{r}'}\, d\tau' \quad -(17)$$