260(6): Beltrami Equation for Momentum and the General Schroedinger Equation.

The problem is to reduce the Beltrami equation for momentum:

$$\underline{\nabla} \times \underline{p} = \kappa \underline{p} \quad -(1)$$

to the Schroedinger equation:

$$(\nabla^2 + \kappa^2)\psi = 0 \quad -(2)$$

where

$$\kappa^2 = \frac{2m}{\hbar^2}(V - E) \quad -(3)$$

Here

$$E = T + V \quad -(4)$$

$$= \frac{p^2}{2m} + V$$

$$E\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi \quad -(5)$$

$$= H\psi$$

It is seen that κ is r dependent in general for the Coulomb potential V or for the general potential κ depends on (r, θ, ϕ).

From eq. (1):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{p}) = \underline{\nabla} \times (\kappa \underline{p}) \quad -(6)$$

2) By vector analysis:

$$\underline{\nabla}(\underline{\nabla}\cdot\underline{p}) - \nabla^2\underline{p} = \kappa\,\underline{\nabla}\times\underline{p} + \underline{\nabla}\kappa\times\underline{p} \quad -(7)$$

$$= \kappa^2\underline{p} + \underline{\nabla}\kappa\times\underline{p}$$

so $$(\nabla^2+\kappa^2)\underline{p} = \underline{\nabla}(\underline{\nabla}\cdot\underline{p}) - \underline{\nabla}\kappa\times\underline{p} \quad -(8)$$

One possible solution is:

$$(\nabla^2+\kappa^2)\underline{p} = 0 \quad -(9)$$

and $$\underline{\nabla}(\underline{\nabla}\cdot\underline{p}) = \underline{\nabla}\kappa\times\underline{p} \quad -(10)$$

$$-(11)$$

Eq. (10) implies:

$$\underline{p}\cdot\underline{\nabla}(\underline{\nabla}\cdot\underline{p}) = \underline{p}\cdot\underline{\nabla}\kappa\times\underline{p} = 0$$

Now use:

$$\underline{p} \rightarrow -\hbar\underline{\nabla} \quad -(12)$$

Eq. (11) becomes:

$$-\hbar^2\underline{\nabla}\cdot\left(\underline{\nabla}(\underline{\nabla}\cdot\underline{\nabla}\phi)\right) = 0 \quad -(13)$$

so $$\underline{\nabla}\cdot(\underline{\nabla}\nabla^2\phi) = 0 \quad -(14)$$

i.e. $$\boxed{\nabla^4\phi = 0} \quad -(15)$$

3) Eqs. (9) and (12) give:

$$(\nabla^2 + \kappa^2)\underline{\nabla}\phi = \underline{0} \quad -(16)$$

Therefore eq. (1) gives:

$$(\nabla^2 + \kappa^2)\underline{\nabla}\phi = 0 \quad -(17)$$

provided that

$$\nabla^4 \phi = 0 \quad -(18)$$

the Schroedinger equation is:

$$(\nabla^2 + \kappa^2)\phi = 0 \quad -(19)$$

From eq. (19):

$$\underline{\nabla}\left((\nabla^2 + \kappa^2)\phi\right) = 0 \quad -(20)$$

i.e.

$$(\nabla^2 + \kappa^2)\underline{\nabla}\phi + \left(\underline{\nabla}(\nabla^2 + \kappa^2)\right)\phi = 0 \quad -(21)$$

A possible solution of eq. (21) is:

$$(\nabla^2 + \kappa^2)\underline{\nabla}\phi = \underline{0} \quad -(22)$$

and

$$\left(\underline{\nabla}(\nabla^2 + \kappa^2)\right)\phi = \underline{0} \quad -(23)$$

Eq. (22) is eq. (17), QED.

4) Eq. (23) can be written as:

$$\underline{\nabla} \nabla^2 \phi + \underline{\nabla} \kappa^2 \phi = \underline{0} \quad -(24)$$

i.e. $$\underline{\nabla} \left(\nabla^2 \phi + \kappa^2 \phi \right) = \underline{0} \quad -(25)$$

A possible solution of eq. (25) is the Schroedinger equation:

$$\left(\nabla^2 + \kappa^2 \right) \phi = 0 \quad -(26)$$

So the Schroedinger equation (26) is compatible with

$$\left(\nabla^2 + \kappa^2 \right) \underline{\nabla} \phi = \underline{0} \quad -(27)$$

the Beltrami equation (1) gives eq. (27) and the constraint

$$\nabla^4 \phi = 0 \quad -(28)$$

From eq. (27):

$$\underline{\nabla} \cdot \left(\left(\nabla^2 + \kappa^2 \right) \underline{\nabla} \phi \right) = 0 \quad -(29)$$

i.e. $$\underline{\nabla} \cdot \left(\nabla^2 \underline{\nabla} \phi + \kappa^2 \underline{\nabla} \phi \right) = 0 \quad -(30)$$

i.e. $$\left(\nabla^2 + \kappa^2 \right) \nabla^2 \phi + \left(\underline{\nabla} \kappa^2 \right) \cdot \underline{\nabla} \phi = 0 \quad -(31)$$

Therefore the constraint (28) together with

5) eq. (31) leads to:

$$\kappa^2 \nabla^2 \phi + \underline{\nabla} \kappa^2 \cdot \underline{\nabla} \phi = 0 \quad -(32)$$

Conclusion

In general the Beltrami equation:

$$\underline{\nabla} \times \underline{p} = \kappa \underline{p} \quad -(33)$$

quantizes to:

$$(\nabla^2 + \kappa^2)\phi = 0 \quad -(34)$$

and

$$(\nabla^2 + \kappa^2)\underline{\nabla}\phi = 0 \quad -(35)$$

provided

$$\nabla^4 \phi = 0 \quad -(36)$$

where

$$\kappa^2 = \frac{2m}{\hbar^2}(V - E). \quad -(37)$$

In order to apply these results to the structure of an electron, proton or neutron, scattering theory is needed. This will be the subject of the next note.