1) 360(3): Conditions under which the Tetrad is a Beltrami Function.

In general the tetrad postulate is:

$$D_\mu q^a_\nu = 0 \quad -(1)$$

and in the absence of a magnetic monopole its space part is:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = 0 \quad -(2)$$

as shown in recent papers. Eq. (2) implies:

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b = \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(3)$$

Eq. (1) can be rewritten as:

$$(\Box + \kappa^2) q^a_\mu = 0 \quad -(4)$$

where

$$q^a_\mu = (q^a_0, -\underline{q}^a) \quad -(5)$$

so

$$(\Box + \kappa^2) q^a_0 = 0 \quad -(6)$$

and

$$(\Box + \kappa^2) \underline{q}^a = \underline{0} \quad -(7)$$

where

$$\kappa^2 = q^\nu_a \partial^\mu (\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}) \quad -(8)$$

and

$$\Box = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2 \quad -(9)$$

Eq. (2) can always be written as:

d)

$$\underline{\nabla} \times (\underline{\omega}^a{}_b \times \underline{q}^b) = K_0\, \underline{\omega}^a{}_b \times \underline{q}^b \quad -(10)$$

where K_0 is a constant. To see this write:

$$\underline{V}^a = \underline{\omega}^a{}_b \times \underline{q}^b \quad -(11)$$

so

$$\underline{\nabla} \times \underline{V}^a = K_0\, \underline{V}^a \quad -(12)$$

It follows that:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{V}^a = K_0\, \underline{\nabla} \cdot \underline{V}^a = 0 \quad -(13)$$

i.e

$$\underline{\nabla} \cdot \underline{V}^a = \underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = 0 \quad -(14)$$

QED. The torsion vector is defined by:

$$\underset{\sim}{T}^a = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \quad -(15)$$

from the first Cartan structure equation. It follows immediately from eq. (2) that:

$$\underline{\nabla} \cdot \underset{\sim}{T}^a = 0. \quad -(16)$$

More accurately:

$$\underline{T}^a(\text{spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \quad -(17)$$

So:

3)

$$\underline{\nabla} \cdot \underset{\sim}{T}^a(\text{spin}) = 0 \quad -(18)$$

so the spin torsion is a Beltrami function:

$$\boxed{\underline{\nabla} \times \underset{\sim}{T}^a(\text{spin}) = \kappa_0 \underset{\sim}{T}^a(\text{spin})} \quad -(19)$$

where κ_0 is a constant.

Note carefully that the orbital torsion is not a Beltrami function in general.

From eqs. (18) and (19):

$$\boxed{\left(\nabla^2 + \kappa_0^2\right) \underset{\sim}{T}^a(\text{spin}) = \underline{0}} \quad -(20)$$

which is a Helmholtz equation with many interesting solutions for $\underset{\sim}{T}^a(\text{spin})$.

In ECE electrodynamics eq. (19) translates to:

$$\underline{\nabla} \times \underline{B}^a = \kappa_0 \underline{B}^a. \quad -(21)$$

From eq. (15):

$$\underline{\nabla} \times \underset{\sim}{T}^a(\text{spin}) = \underline{\nabla} \times (\underline{\nabla} \times \underline{q}^a) - \underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{q}^b\right) \quad -(22)$$

and using eq. (10):

4)

$$\underline{\nabla} \times \underset{\sim}{T}^a(\text{spin}) = \underline{\nabla} \times (\underline{\nabla} \times \underline{q}^a) - \kappa_0 \underline{\omega}^a{}_b \times \underline{q}^b$$

$$= \kappa_0 \underset{\sim}{T}^a(\text{spin}) \quad -(23)$$

$$= \kappa_0 \left(\underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \right).$$

so

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{q}^a) = \kappa_0 \underline{\nabla} \times \underline{q}^a \quad -(24)$$

and

$$\boxed{\underline{\nabla} \times \underline{q}^a = \kappa_0 \underline{q}^a} \quad -(25)$$

The space part of the tetrad is a Beltrami function. In ECE electrodynamics eq. (25) translates to:

$$\underline{\nabla} \times \underline{A}^a = \kappa_0 \underline{A}^a. \quad -(26)$$

It follows that:

$$\boxed{\underline{\nabla} \cdot \underline{q}^a = 0} \quad -(27)$$

and

$$\boxed{(\nabla^2 + \kappa_0^2)\, \underline{q}^a = \underline{0}} \quad -(28)$$

The space part of the tetrad four-vector obeys the Helmholtz wave equation.

) from eqs. (3) and (25) it follows that:

$$\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa_0 \, \underline{\omega}^a{}_b \quad -(29)$$

and

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b = 0 \quad -(30)$$

so

$$\boxed{(\nabla^2 + \kappa_0^2)\, \underline{\omega}^a{}_b = \underline{0}} \quad -(31)$$

The spin connection obeys a Helmholtz wave equation in the absence of a magnetic monopole.

Therefore:

$$(\Box + \kappa^2)\, \underline{q}^a = \underline{0} \quad -(32)$$

$$(\nabla^2 + \kappa_0^2)\, \underline{q}^a = \underline{0} \quad -(33)$$

Eq. (32) is:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \kappa^2 \right) \underline{q}^a = \underline{0} \quad -(34)$$

so

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \kappa_0^2 + \kappa^2 \right) \underline{q}^a = \underline{0} \quad -(35)$$

Therefore:

$$\boxed{\underline{q}^a = \underline{q}^a(\underline{r}) \exp(-i\omega t)} \quad -(36)$$

6) where:

$$\boxed{\omega = c\left(\kappa_0^2 + \kappa^2\right)^{1/2}} \quad -(37)$$

and
$$\underline{\nabla} \times \underline{q}^a(\underline{r}) = \kappa_0 \, \underline{q}^a(\underline{r}) \quad -(38)$$

It follows that:

$$\left(\Box + \kappa^2\right) \underline{q}^a = \underline{0} \quad -(39)$$

and
$$\left(\nabla^2 + \kappa_0^2\right) \underline{q}^a(\underline{r}) = \underline{0} \quad -(40)$$

where
$$\underline{q}^a = \underline{q}^a(\underline{r}) \exp(-i\omega t) \quad -(41)$$

The space part of the tetrad, $\underline{q}^a(\underline{r})$, the complete obeys a Helmholtz equation and tetrad, $\underline{q}^a$, obeys a d'Alembert equation. Eq. (40) is non-relativistic and eq. (39) is relativistic. They lead to the Schroedinger and fermion equations respectively.

The complete tetrad also obeys a Beltrami and Helmholtz equation.

1) A self consistency check may be carried out using the vector identity:

$$\underline{\nabla} \times (\underline{\omega}^a{}_b \times \underline{q}^b) = \underline{\omega}^a{}_b (\underline{\nabla} \cdot \underline{q}^b) - (\underline{\nabla} \cdot \underline{\omega}^a{}_b)\underline{q}^b + (\underline{q}^b \cdot \underline{\nabla})\underline{\omega}^a{}_b - (\underline{\omega}^a{}_b \cdot \underline{\nabla})\underline{q}^b \quad -(42)$$

It follows that

$$\boxed{\kappa_0\, \underline{\omega}^a{}_b \times \underline{q}^b = (\underline{q}^b \cdot \underline{\nabla})\underline{\omega}^a{}_b - (\underline{\omega}^a{}_b \cdot \underline{\nabla})\underline{q}^b} \quad -(43)$$

and

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = \underline{\nabla} \cdot \left((\underline{q}^b \cdot \underline{\nabla})\underline{\omega}^a{}_b - (\underline{\omega}^a{}_b \cdot \underline{\nabla})\underline{q}^b \right) \quad -(44)$$

$$= 0$$

In these equations:

$$(\underline{q}^b \cdot \underline{\nabla})\underline{\omega}^a{}_b = q^b_X \frac{\partial \omega^a{}_{Xb}}{\partial X} + \cdots \quad -(45)$$

and

$$\underline{\nabla} \cdot \left((\underline{q}^b \cdot \underline{\nabla})\underline{\omega}^a{}_b \right) = \frac{\partial}{\partial X}\left(q^b_X \frac{\partial \omega^a{}_{Xb}}{\partial X} \right) + \cdots \quad -(46)$$

$$= \frac{\partial q^b_X}{\partial X}\frac{\partial \omega^a{}_{Xb}}{\partial X} + q^b_X \frac{\partial^2 \omega^a{}_{Xb}}{\partial X^2} + \cdots$$

Similarly:

$$(\underline{\omega}^a{}_b \cdot \underline{\nabla})\underline{q}^b = \omega^a{}_{xb} \frac{\partial q^b_x}{\partial X} + \cdots \quad -(47)$$

and:

$$\underline{\nabla} \cdot ((\underline{\omega}^a{}_b \cdot \underline{\nabla})\underline{q}^b) = \frac{\partial}{\partial X}\left(\omega^a{}_{xb} \frac{\partial q^b_x}{\partial X}\right) + \cdots$$

$$= \frac{\partial \omega^a{}_{xb}}{\partial X}\frac{\partial q^b_x}{\partial X} + \omega^a{}_{xb}\frac{\partial^2 q^b_x}{\partial X^2} + \cdots \quad -(48)$$

From eqs. (44), (46) and (48):

$$q^b_x \frac{\partial^2 \omega^a{}_{xb}}{\partial X^2} + \cdots = \omega^a{}_{xb} \frac{\partial^2 q^b_x}{\partial X^2} + \cdots \quad -(49)$$

i.e.

$$\underline{q}^b \cdot \underline{\nabla}(\underline{\nabla} \cdot \underline{\omega}^a{}_b) = \underline{\omega}^a{}_b \cdot \underline{\nabla}(\underline{\nabla} \cdot \underline{q}^b) \quad -(50)$$

This is true because:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b = 0 \quad -(51)$$

and

$$\underline{\nabla} \cdot \underline{q}^b = 0 \quad -(52)$$

QED