257(5): General Solution in Terms of Bessel Functions

As discussed by Reed in his section 15, the general solution of

$$\underline{\nabla} \times \underline{B} = k\underline{B} \quad -(1)$$

with

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(2)$$

is the Helmholtz equation:

$$\nabla^2 \underline{B} + k^2 \underline{B} = 0 \quad -(3)$$

in terms of

$$\nabla^2 \phi + k^2 \phi = 0 \quad -(4)$$

From eq. (4) the solutions of (1) are:

$$\underline{L} = \underline{\nabla} \phi \quad -(5)$$

$$\underline{P} = \underline{\nabla} \times (\phi \underline{a}) \quad -(6)$$

$$\underline{T} = \underline{\nabla} \times \underline{P} \quad -(7)$$

where $\underline{a}$ is an arbitrary constant vector. In general,

$$\underline{B} = B \left(k \underline{\nabla} \times (\phi \underline{a}) + \underline{\nabla} \times \underline{\nabla} \times (\phi \underline{a}) \right) \quad -(8)$$

$$= B^{(0)} \left(k \underline{P} + \underline{T} \right) \quad -(9)$$

where $\underline{P}$ is the poloidal and $\underline{T}$ the toroidal solution. For a vector potential of type:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)} \quad -(10)$$

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(11)$$

2) and
$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}^{(1)}) = \kappa^2 \underline{A}^{(1)} \quad -(12)$$
$$= \kappa \underline{\nabla} \times \underline{A}^{(1)}$$

so
$$\underline{B}^{(1)} = \frac{1}{2} \left(\underline{\nabla} \times \underline{A}^{(1)} + \frac{1}{\kappa} \underline{\nabla} \times (\underline{\nabla} \times \underline{A}^{(1)}) \right) \quad -(13)$$

Eqs. (8) and (13) are the same if:
$$k = \kappa, \quad \underline{A}^{(1)} = \phi \underline{a}, \quad B = 1/(2\kappa) \quad -(14)$$
Reed's eq. (34):
The general solution of eq. (8) is
$$\underline{B} = \sum_{m,n} B_{mn} \underline{b}_{mn}(r, \theta, Z) \quad -(15)$$
where the Helmholtz equation reduces to an axisymmetric wave equation:
$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + k^2 \phi = 0 \quad -(16)$$
then
$$\phi = C J_0(kr) \quad -(17)$$
where C is a constant and J_0 the zero order Bessel function. If:
$$m = 0, \quad n = 0, \quad \underline{a} = (0, 0, 1) \quad -(18)$$
then:
$$\underline{B} = B^{(0)} (0, J_1(kr), J_0(kr)) \quad -(19)$$
in cylindrical polar coordinates. So:

3)

$$\underline{B} = B^{(0)}\left(J_1(kr)\, \underline{e}_\theta + J_0(kr)\, \underline{k} \right) \quad -(20)$$

where $\underline{e}_\theta$ is the unit vector:

$$\underline{e}_\theta = -\sin\theta\, \underline{i} + \cos\theta\, \underline{j} \quad -(21)$$

Reed's Figure (3) is obtained from eq. (20).

If

$$x = kr \quad -(22)$$

the Bessel functions are defined by:

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(-x\sin(\tau))\, d\tau$$

$$= \frac{1}{\pi}\int_0^\pi \cos(x\sin(\tau))\, d\tau \quad -(23)$$

and

$$J_1(x) = \frac{1}{\pi}\int_0^\pi \cos(\tau - x\sin(\tau))\, d\tau \quad -(24)$$

The $\underline{B}^{(3)}$ type solution is:

$$\underline{B} \rightarrow B^{(0)} J_0(kr)\, \underline{k} \quad -(25)$$

and as $r \rightarrow \infty$ approaches infinitesimal oscillations about the $\underline{k}$ axis.