257(6): Relation between Free Field Equations and Beltrami Equations.

The plane wave solutions of the free field equations:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(1)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(2)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(3)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0} \quad -(4)$$

are also solutions of the Beltrami equations:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad -(5)$$

$$\underline{\nabla} \times \underline{E} = \kappa \underline{E} \quad -(6)$$

For example if:

$$\underline{E} = \frac{E^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(7)$$

then

$$\underline{B} = \frac{B^{(0)}}{\sqrt{2}} (i\underline{i} + \underline{j}) e^{i\phi} \quad -(8)$$

where

$$\phi = \omega t - \kappa Z \quad -(9)$$

and

$$E^{(0)} = c B^{(0)} \quad -(10)$$

Here ω is the angular frequency at instant t, κ is the wave vector magnitude at Z, $\underline{B}$ is the magnetic flux density, $\underline{E}$ is the electric field strength, and c is the vacuum speed of light. Note carefully that for identically non-zero photon mass

1) then c is not the speed of light, it is a constant agreed upon in standards laboratories.

The free field equations are Beltrami equations in vacuo.

This proven as follows. From eq. (4):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) - \frac{1}{c^2} \underline{\nabla} \times \frac{\partial \underline{E}}{\partial t} = \underline{0} \,. \quad -(11)$$

Fm eq. (3):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) + \underline{\nabla} \times \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(12)$$

Now use:

$$\underline{\nabla} \times \frac{\partial \underline{E}}{\partial t} = \frac{\partial}{\partial t} (\underline{\nabla} \times \underline{E}) \quad -(13)$$

$$\underline{\nabla} \times \frac{\partial \underline{B}}{\partial t} = \frac{\partial}{\partial t} (\underline{\nabla} \times \underline{B}) \quad -(14)$$

to find:

$$\boxed{\begin{aligned} \underline{\nabla} \times (\underline{\nabla} \times \underline{B}) &= \frac{1}{c^2} \frac{\partial}{\partial t} \underline{\nabla} \times \underline{E} \quad -(15) \\ \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) &= - \frac{\partial}{\partial t} \underline{\nabla} \times \underline{B} \quad -(16) \end{aligned}}$$

Fm eqs (5) and (6):

$$\kappa^2 \underline{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \kappa \underline{E} \quad -(17)$$

i.e

$$\frac{\partial \underline{E}}{\partial t} = c^2 \kappa \underline{B} = \omega c \underline{B} \quad -(18)$$

) and

$$\kappa^2 \underline{E} + \kappa \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad - (19)$$

i.e.

$$\frac{\partial \underline{B}}{\partial t} = -\frac{\omega}{c} \underline{E} \quad - (20)$$

The solutions of eqns. (18) and (20) are eqns. (7) and (8), QED.

Note that:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{B}) - \nabla^2 \underline{B} \quad - (21)$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{E}) - \nabla^2 \underline{E} \quad - (22)$$

For the free field, eqs. (1) and (2) apply, so:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = -\nabla^2 \underline{B} = \kappa^2 \underline{B} \quad - (23)$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\nabla^2 \underline{E} = \kappa^2 \underline{E} \quad - (24)$$

and we obtain the Helmholtz wave equations:

$$\nabla^2 \underline{B} + \kappa^2 \underline{B} = \underline{0} \quad - (25)$$

$$\nabla^2 \underline{E} + \kappa^2 \underline{E} = \underline{0}. \quad - (26)$$

These are the Beltrami equations:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = \kappa^2 \underline{B} \quad - (27)$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \kappa^2 \underline{E} \quad - (28)$$

So solutions of the Beltrami equations are also solutions of the Helmholtz wave equations and are also solutions of the Maxwell Heaviside equations.

4) From eqs. (11), (27) and (20):

$$-\nabla^2 \underline{B} - \frac{1}{c^2}\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{E})$$

$$= -\nabla^2 \underline{B} - \frac{\kappa}{c^2}\frac{\partial E}{\partial t}$$

$$= -\nabla^2 \underline{B} + \frac{c}{\omega}\frac{\kappa}{c^2}\frac{\partial^2 \underline{B}}{\partial t^2} \quad -(29)$$

$$= -\nabla^2 \underline{B} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{B} = \underline{0}$$

which is the d'Alembert equation

$$\square \underline{B} = \underline{0} \quad -(30)$$

For finite photon mass m_0:

$$\hbar^2\omega^2 = c^2\hbar^2\kappa^2 + m_0^2 c^4 \quad -(31)$$

in which case:

$$\left(\square + \left(\frac{m_0 c}{\hbar}\right)^2\right)\underline{B} = 0 \quad -(31)$$

which is the Proca equation. Note that the phase in eq (31) is:

$$\phi = \omega t - \kappa Z \quad -(32)$$

with:

$$\frac{\omega^2}{c^2} = \kappa^2 + \left(\frac{m_0 c}{\hbar}\right)^2 \quad -(33)$$

From eqs. (15) and (16):

$$\kappa \, \underline{\nabla} \times \underline{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \underline{\nabla} \times \underline{E} \quad - (34)$$

$$\kappa \, \underline{\nabla} \times \underline{E} = - \frac{\partial}{\partial t} \underline{\nabla} \times \underline{B} \quad - (35)$$

Therefore:

$$\frac{\partial^2}{\partial t^2} \underline{\nabla} \times \underline{B} = -\omega^2 \underline{\nabla} \times \underline{B} \quad - (36)$$

$$\frac{\partial^2}{\partial t^2} \underline{\nabla} \times \underline{E} = -\omega^2 \underline{\nabla} \times \underline{E} \quad - (37)$$

In general:

$$\frac{\partial}{\partial t^2} e^{i(\omega t - \kappa Z)} = -\omega^2 e^{i(\omega t - \kappa Z)} \quad - (38)$$

and

$$e^{i(\omega t - \kappa Z)} = e^{i\omega t} e^{-i\kappa Z} \quad - (39)$$

so the general solution of the Beltrami equation:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad - (40)$$

will also be a solution of the Maxwell Heaviside field equations (1) to (4) provided that the solution takes the form:

$$\underline{B}_1 = e^{i\omega t} \underline{B} \quad - (41)$$

and

$$\underline{E}_1 = e^{i\omega t} \underline{E} \quad - (42)$$

6) As discussed by Reed in his section 15, the general solution of:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad -(43)$$

is

$$\underline{B} = B^{(0)} \left(\kappa \underline{\nabla} \times (\phi \underline{a}) + \underline{\nabla} \times \underline{\nabla} \times (\phi \underline{a}) \right) \quad -(44)$$

in which $\underline{a}$ is an arbitrary constant vector and

$$\nabla^2 \phi + \kappa^2 \phi = 0 \quad -(45)$$

Here the poloidal solution is:

$$\underline{P} = \underline{\nabla} \times (\phi \underline{a}) \quad -(46)$$

and the toroidal solution is:

$$\underline{T} = \underline{\nabla} \times \underline{P} \quad -(47)$$

Here:

$$\underline{\nabla} \times (\phi \underline{a}) = \phi \underline{\nabla} \times \underline{a} + (\underline{\nabla} \phi) \times \underline{a} \quad -(48)$$

therefore the solution of the MH field equation corresponding to eq. (44) is:

$$\boxed{\underline{B} = B^{(0)} e^{i\omega t} \left(\kappa \underline{\nabla} \times (\phi \underline{a}) + \underline{\nabla} \times \left(\underline{\nabla} \times (\phi \underline{a}) \right) \right)} \quad -(49)$$

1) As discussed by Reed on his page 546 ff eq. (49) represents spiral field lines that lie on an axisymmetric torus surface, with the pitch of the spirals changing gradually from totally poloidal to completely toroidal.

The general solution of the Beltrami equation in cylindrical polar coordinates is:

$$\underline{B} = \sum_{m,n} B_{mn}^{(0)} \underline{B}_{mn}(r, \theta, Z) \quad -(50)$$

where m is a non-negative integer and where the modes $\underline{B}_{mn}$ depend on θ and Z through a phase function $m\theta + nZ$. These modes are combinations of the Bessel and Neumann functions. Eq. (50) is also a solution of the MH equations (1) to (4) if:

$$\boxed{\underline{B} = e^{i\omega t} \sum_{m,n} B_{mn}^{(0)} \underline{B}_{mn}(r, \theta, Z) \quad -(51)}$$

when eq. (45) simplifies to:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + k^2\phi = 0 \quad -(52)$$

Then:

$$\phi = C J_0(\kappa r) \quad -(53)$$

where J_0 is the Bessel function of order 0 and where C is a constant. Then for mode:

$$m = 0, \; n = 0, \; \underline{a} = (0, 0, 1) \quad -(54)$$

the solution of the free field MH equations is:

$$\underline{B} = B^{(0)} e^{i\omega t} \left(0, J_1(\kappa r), J_0(\kappa r)\right) \quad -(55)$$

$$\boxed{\underline{B} = B^{(0)} e^{i\omega t} \left(J_1(\kappa r) \underline{e}_\theta + J_0(\kappa r) \underline{k} \right)} \quad -(56)$$

Reference

1) H. Zaghloul and O. Barajas, Am. J. Phys., 58, 783 (1990)
2) G. E. Marsh, "Force Free Magnetic Fields" (1996).
3) H. Alfven, "Cosmical Electrodynamics" (Oxford, 1963).

Eq. (56) is a standard vortex solution for the free electromagnetic field sketched in Reed's fig (3). Here:

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(57)$$

The Bessel functions are:

$$J_0(\kappa r) = \frac{1}{\pi} \int_0^\pi \cos(\kappa r \sin(\tau)) \, d\tau \quad -(58)$$

and:

$$J_1(kr) = \frac{1}{\pi}\int_0^{\pi} \cos(\tau - kr\sin(\tau))\, d\tau \quad -(59)$$

<u>Longitudinal Component</u>

$$B_z = B^{(0)} e^{i\omega t} J_0(kr) \quad -(60)$$

$$= B^{(0)} \frac{e^{i\omega t}}{\pi}\int_0^{\pi} \cos(kr\sin(\tau))\, d\tau$$

<u>Transverse Component</u>

$$\underline{B}_\theta = B^{(0)} e^{i\omega t} J_1(kr)\, \underline{e}_\theta \quad -(61)$$

$$= B^{(0)} \frac{e^{i\omega t}}{\pi}\int_0^{\pi} \cos(\tau - kr\sin(\tau))\, d\tau\, (-\sin\theta\, \underline{i} + \cos\theta\, \underline{j})$$

These can be expressed as:

$$B_z = B^{(0)} \frac{e^{i\omega t}}{2\pi}\int_0^{\pi} e^{ix} + e^{-ix}\, d\tau \quad -(62)$$

$$= \frac{B^{(0)}}{2\pi}\int_0^{\pi} e^{i(\omega t + x)} + e^{i(\omega t - x)}\, d\tau$$

where

$$x = kr\sin\tau \quad -(63)$$

10) and

$$\underline{B}_\theta = \frac{B^{(0)}}{2\pi}\int_0^\pi e^{i\omega t}\left(e^{iy}+e^{-iy}\right)d\tau\left(-\sin\theta\,\underline{i}+\cos\theta\,\underline{j}\right)$$

$$= \frac{B^{(0)}}{2\pi}\int_0^\pi\left(e^{i(\omega t+y)}+e^{i(\omega t-y)}\right)d\tau\left(-\sin\theta\,\underline{i}+\cos\theta\,\underline{j}\right) \quad -(64)$$

where

$$y = \tau - \kappa r \sin\tau$$

These have been observed experimentally in plasma physics:

1) D. R. Wells, Phys. Fluids, 7, 826 (1964).
2) D. R. Wells, Int. J. Fusion Energy, 3, 17 (1985).

H. Alfven has also shown that they are responsible for spiral arms in galaxies.

This note has shown that they are also solutions of the free field MH equations, and that the longitudinal component indicates photon mass.

Now use:

$$\sin\theta = \frac{1}{i}\sinh(i\theta) = -\frac{i}{2}\left(e^{i\theta}-e^{-i\theta}\right) \quad -(65)$$

and

11)

$$\cos\theta = \cosh(i\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad -(66)$$

to express the transverse component in the form of plane waves:

$$\underline{B}^{(1)}_{\theta} = \frac{B^{(0)}}{4\pi}\int_0^{\pi}\Big[i\,\underline{i}\left(e^{i(\omega t+y)} + e^{i(\omega t-y)}\right)\left(e^{i\theta} - e^{-i\theta}\right) + \underline{j}\left(e^{i(\omega t+y)} + e^{i(\omega t-y)}\right)\left(e^{i\theta} + e^{-i\theta}\right)\Big] d\tau \quad -(67)$$

and

$$\underline{B}^{(2)}_{\theta} = \frac{B^{(0)}}{4\pi}\int_0^{\pi}\Big[-i\,\underline{i}\left(e^{-i(\omega t+y)} + e^{-i(\omega t-y)}\right)\left(e^{-i\theta} - e^{i\theta}\right) + \underline{j}\left(e^{-i(\omega t+y)} + e^{-i(\omega t-y)}\right)\left(e^{-i\theta} + e^{i\theta}\right)\Big] d\tau \quad -(68)$$

and $\underline{B}^{(3)}$ type fields emerge from:

$$\underline{B}^{(3)} = -\frac{i}{B^{(0)}}\,\underline{B}^{(1)}_{\theta} \times \underline{B}^{(2)}_{\theta} \quad -(69)$$

For example:

$$\underline{B}_\theta^{(1)} = \frac{B^{(0)}}{4\pi} \int_0^{\pi} (i\underline{i} + \underline{j}) e^{-i(\omega t + y + \theta)} d\tau \quad -(70)$$

$$+ \dots$$

$$\underline{B}_\theta^{(2)} = \frac{B^{(0)}}{4\pi} \int_0^{\pi} (-i\underline{i} + \underline{j}) e^{-i(\omega t + y + \theta)} d\tau \quad -(71)$$

$$+ \dots$$

and:

$$\underline{B}_\theta^{(1)} \times \underline{B}_\theta^{(2)} = \left[\frac{B^{(0)2}}{8\pi^2} i \int_0^{\pi} e^{iy} d\tau \int_0^{\pi} e^{-iy} d\tau \right] \underline{k} \quad -(72)$$

where

$$y = \tau - \kappa r \sin \tau .$$

In eq. (72):

$$e^{iy} = \cos y + i \sin y \quad -(73)$$

$$e^{-iy} = \cos y - i \sin y \quad -(74)$$

so

$$\int_0^{\pi} e^{iy} d\tau = f_1(y) + i f_2(y) \quad -(75)$$

$$\int_0^{\pi} e^{-iy} d\tau = f_1(y) - i f_2(y) \quad -(76)$$

It follows that:

$$\underline{B}_\theta^{(1)} \times \underline{B}_\theta^{(2)} = \frac{i B^{(0)2}}{8\pi^2} \underline{k} \quad -(77)$$

13) and the $\underline{B}^{(3)}$ field is:

$$\boxed{\underline{B}^{(3)} = -\frac{B^{(0)}}{8\pi^2} \cdot \underline{k}} \quad -(78)$$

Conclusions

1) Solutions to the general Beltrami equation are also solutions of the free electromagnetic field equations provided they are multiplied by $\exp(i\omega t)$.
2) The ~~sheared~~ vortex solution (56) contains a longitudinal component, indicating photon mass.
3) The ~~sheared~~ vortex solution can give rise to plane waves and $\underline{B}^{(3)}$ type solutions.
4) The arms of whirlpool galaxies can be thought of in terms of a Beltrami equation, according to Alfven, "Cosmical Electrodynamics" (Clarendon, Oxford, 1963, second edition), so the jet energy from a whirlpool galaxy indicates the presence of $\underline{B}^{(3)}$ type structures.