257(4): General Solution of the Beltrami Equation in Terms of Elliptically Polarized Plane Waves.

The general solution can be written in terms of:

$$B_X = B_2 \sin \phi_Z + B_3 \cos \phi_Y \quad -(1)$$

$$B_Y = B_3 \sin \phi_X + B_1 \cos \phi_Z \quad -(2)$$

$$B_Z = B_1 \sin \phi_Y + B_2 \cos \phi_X \quad -(3)$$

so

$$\underline{B} = B_X \underline{i} + B_Y \underline{j} + B_Z \underline{k} \quad -(4)$$

and

$$\underline{\nabla} \times \underline{B} = \lambda \underline{B} \quad -(5)$$

where λ is a constant. Eqns. (1) to (3) have the structure of ABC flow:

$$u_1 = A \sin z + C \cos y \quad -(6)$$

$$u_2 = B \sin x + A \cos z \quad -(7)$$

$$u_3 = C \sin y + B \cos x \quad -(8)$$

These are examples of 3D chaotic phenomena and Lagrangian turbulence.

Now use:

$$\cos \phi = \frac{1}{2} \left(e^{i\phi} + e^{-i\phi} \right) \quad -(9)$$

$$\sin \phi = -\frac{i}{2} \left(e^{i\phi} - e^{-i\phi} \right) \quad -(10)$$

and eq. (4) can be written as a sum of three terms as follows:

$$
\begin{aligned}
\underline{B} = {} & B_2 \sin\phi_Z \, \underline{i} + B_1 \cos\phi_Z \, \underline{j} \\
& + B_3 \cos\phi_Y \, \underline{i} + B_1 \sin\phi_Y \, \underline{k} \\
& + B_3 \sin\phi_X \, \underline{j} + B_2 \cos\phi_X \, \underline{k}
\end{aligned}
\quad -(11)
$$

$$
\begin{aligned}
= \frac{1}{2} \Big[& (i\underline{i} B_2 + B_1 \underline{j}) e^{-i\phi_Z} + (-i\underline{i} B_2 + B_1 \underline{j}) e^{i\phi_Z} \\
& + (B_3 \underline{i} - i B_1 \underline{k}) e^{i\phi_Y} + (B_3 \underline{i} + i B_1 \underline{k}) e^{-i\phi_Y} \\
& + (B_2 \underline{k} - i B_3 \underline{j}) e^{i\phi_X} + (B_2 \underline{k} + i B_3 \underline{j}) e^{-i\phi_X} \Big]
\end{aligned}
$$

If the phases are defined as:

$$\phi_X = \omega_X t - \kappa_X X \quad -(12)$$

$$\phi_Y = \omega_Y t - \kappa_Y Y \quad -(13)$$

$$\phi_Z = \omega_Z t - \kappa_Z Z \quad -(14)$$

Eq. (11) represents a sum of three elliptically polarized waves in X, Y and Z. Each of the six terms in eq. (11) obey a Beltrami equation and are also solutions to the vacuum equations of the simplified ECE engineering model. There are also several conjugate products possible among the terms

3) For example, if:

$$\underline{B}^{(1)} = \left(iB_2\,\underline{i} + B_1\,\underline{j}\right)e^{-i\phi_2} = \underline{B}^{(2)*} \quad -(15)$$

then:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ iB_2 & B_1 & 0 \\ -iB_2 & B_1 & 0 \end{vmatrix} = 2iB_1B_2\,\underline{k}, \quad -(16)$$

giving

$$\underline{B}^{(3)}_{Z} = 2B_1B_2\,\underline{k}. \quad -(17)$$

Similarly

$$\underline{B}^{(3)}_{Y} = 2B_1B_3\,\underline{j} \quad -(18).$$

and

$$\underline{B}^{(3)}_{X} = 2B_2B_3\,\underline{i} \quad -(19)$$

Since these equations have the structure of eqns. (6) to (8), i.e. of ABC flows, a transition to turbulence should occur under suitably defined conditions.

Finally it is possible to consider a plane wave of the type:

$$\underline{B}^{(1)} = \frac{B^{(0)}(Z)}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right)e^{i(\omega t - \kappa Z)} \quad -(20)$$

but this gives a type of non-Beltrami equation:

4)

$$\underline{\nabla} \times \underline{B}^{(1)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ B^{(0)}(Z) e^{i\phi} & -iB^{(0)}(Z) e^{i\phi} & 0 \end{vmatrix} \quad -(21)$$

$$= \kappa \underline{B}^{(1)} + \frac{1}{\sqrt{2}} \frac{\partial B(Z)}{\partial Z} (\underline{i} + i\underline{j}) e^{i\phi}$$

i.e.

$$\underline{\nabla} \times \underline{B}_L^{(1)} = \kappa \underline{B}_L^{(1)} + \frac{\partial B(Z)}{\partial Z} \underline{B}_R^{(1)} \quad -(22)$$

where:

$$\underline{B}_L^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(23)$$

$$\underline{B}_R^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{i\phi} \quad -(24)$$

Therefore the coefficients B_1, B_2 and B_3 must be independent of X, Y and Z.