257(2): Beltrami Electrodynamics and the Spin Connection

In UFT256 it was shown that:

$$\underline{\nabla} \times \underline{q} = \kappa \underline{q} = -i \underline{\omega} \times \underline{q} \quad -(1)$$

where

$$\omega^\mu = (\omega_0, \underline{\omega}) \quad -(2)$$

is the spin connection. It was shown that:

$$\omega^\mu = \kappa^\mu = \left(\frac{\omega}{c}, \underline{\kappa} \right) \quad -(3)$$

so in eq. (1):

$$\kappa = \frac{\omega}{c} \quad -(4)$$

Eq. (1) is an example of Beltrami electrodynamics. Now define:

$$\underline{q}_L^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(5)$$

$$\underline{q}_R^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{i\phi} \quad -(6)$$

$$\underline{q}_L^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} \quad -(7)$$

$$\underline{q}_R^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{-i\phi} \quad -(8)$$

$$\underline{q}^{(3)} = \underline{k} \quad -(9)$$

where the electromagnetic phase ϕ is:

$$\phi = \omega t - \kappa Z \quad -(10)$$

It follows that:

$$\underline{\nabla} \times \underline{q}_L^{(1)} = \kappa \underline{q}_L^{(1)} \quad -(11)$$

$$\underline{\nabla} \times \underline{q}^{(2)}_{L} = \kappa\, \underline{q}^{(2)}_{L} \quad -(12)$$

$$\underline{\nabla} \times \underline{q}^{(1)}_{R} = -\kappa\, \underline{q}^{(1)}_{R} \quad -(13)$$

$$\underline{\nabla} \times \underline{q}^{(2)}_{R} = -\kappa\, \underline{q}^{(2)}_{R} \quad -(14)$$

$$\underline{\nabla} \times \underline{q}^{(3)} = 0\, \underline{q}^{(3)} \quad -(15)$$

The curl eigenvalues are κ, 0, $-\kappa$, and these are the eigenvalues of helicity within a factor κ.

These are examples of the Beltrami equation:

$$\underline{\nabla} \times \underline{v} = \alpha\, \underline{v} \quad -(16)$$

where $\underline{v}$ is velocity and α is a scalar.

References

1) D. Reed in M. W. Evans, ed., "Modern Nonlinear Optics" (Wiley, 2001), second edition, volume 3.

2) H. E. Moses, SIAM J. Appl. Mech., $\underline{21(1)}$, 114 - 144 (1971)

As discussed by Reed, Beltrami flow has no Magnus force. It is a helicoidal flow of the type sketched in Fig (1).

Therefore the equivalent of the $\underline{B}^{(3)}$ field in hydrodynamics

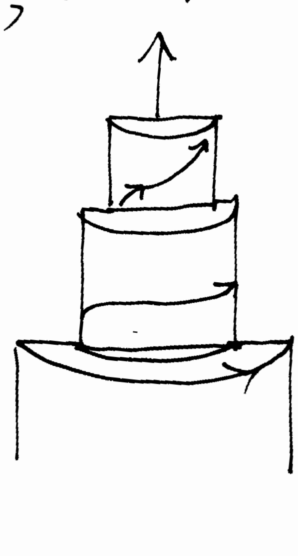

Fig. (1)

) or hydrodynamics is described by the longitudinal line the Beltrami formalism in Fig. (1). In electrodynamics leads to the B Cyclic for vacuum plane waves Theorem:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = i B^{(0)} \underline{B}^{(3)*} \quad -(17)$$

et cyclicum

In flow dynamics:

$$\underline{v}^{(1)} \times \underline{v}^{(2)} = i v^{(0)} \underline{v}^{(3)*} \quad -(18)$$

et cyclicum

In electrodynamics the constant α become the scalar part of the spin connection is a relativistic. Reed shows by application of the Stokes ~~theory~~ Theorem ~~that~~:

$$\alpha = \frac{1}{\pi r^2} \int \theta \, ds = \frac{2\bar{\theta}}{r} \quad -(19)$$

where πr^2 is the area of the flow and $\bar{\theta}$ denotes an average angle. The factor α is the torsion of any Beltrami field neighbouring vector lines $\underline{v}$ in

Similarly, in electrodynamics, the scalar part of the spin connection is the torsion of neighbouring vector lines of potential or magnetic flux density or electric field strength:

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(20)$$

4)

In the simplified version of the engineering model this becomes:

$$\underline{\nabla} \times \underline{A} = \omega_0 \underline{A} \quad -(21)$$

where

$$\omega_0 = |\underline{\omega}| = \kappa, \quad -(22)$$

$$\underline{\omega} = \kappa \underline{k} . \quad -(23)$$

Similarly:

$$\underline{\nabla} \times \underline{B} = \omega_0 \underline{B} \quad -(24)$$

Eqs. (21) and (24) apply not only to vacuum plane waves but to any $\underline{A}$ or $\underline{B}$ that obeys eq. (21) and (24).

Numerical solution of eqs. (21) or (24) will produce a helicoidal flow of the type sketched in Fig. (1), and therefore a $\underline{B}^{(3)}$ field.

The equations of vacuum electromagnetism are:

$$\underline{\omega} \cdot \underline{B} = 0 \quad -(25)$$

$$\underline{\omega} \cdot \underline{E} = 0 \quad -(26)$$

$$\omega_0 \underline{B} - \frac{1}{c} \underline{\omega} \times \underline{E} = \underline{0} \quad -(27)$$

$$\omega_0 \underline{E} + c\, \underline{\omega} \times \underline{B} = \underline{0} \quad -(28)$$

For the transverse plane wave:

$$\underline{\omega} = \omega_0 \underline{k} \quad -(29)$$

and $\underline{E}$ and $\underline{B}$ can be represented by:

$$\underline{B}^{(1)} = \underline{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}} \left(i\underline{i} + \underline{j} \right) e^{i\phi} \quad -(30)$$

$$\underline{E}^{(1)} = \underline{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}} \left(\underline{i} - i\underline{j} \right) e^{i\phi} \quad -(31)$$

It follows that:

$$\underline{\nabla} \times \underline{E}^{(1)} = \kappa \underline{E}^{(1)} \quad -(32)$$

$$\underline{\nabla} \times \underline{B}^{(1)} = \kappa \underline{B}^{(1)} \quad -(33)$$

The longitudinal $\underline{B}^{(3)}$ field is

$$\underline{B}^{(3)} = B^{(0)} \underline{k} \quad -(34)$$

so

$$\underline{\nabla} \times \underline{B}^{(3)} = 0 \, \underline{B}^{(3)} \quad -(35)$$

So for the $\underline{B}^{(3)}$ field:

$$\omega_0 = |\underline{\omega}| = 0 \quad -(36)$$

so:

$$\underline{\omega} \cdot \underline{B}^{(3)} = 0 \quad -(37)$$

$$\omega_0 \underline{B}^{(3)} = \underline{0} \quad -(38)$$

$$\underline{\omega} \times \underline{B}^{(3)} = \underline{0} \quad -(39)$$

For the transverse waves, eqns. (27) and (28) can be combined with eqns. (32) and (33) to give:

6)

$$\omega_0 \underline{B}^{(1)} - \frac{1}{c} \underline{\omega} \times \underline{E}^{(1)} = 0 \quad -(40)$$

$$\omega_0 \underline{E}^{(1)} + c\, \underline{\omega} \times \underline{B}^{(1)} = \underline{0} \quad -(41)$$

Fm eq. (33) ii eq. (40):

$$\frac{\omega_0}{\kappa} \underline{\nabla} \times \underline{B}^{(1)} = \frac{1}{c} \underline{\omega} \times \underline{E}^{(1)} \quad -(42)$$

Fm eq. (32) ii eq. (41):

$$\frac{\omega_0}{\kappa} \underline{\nabla} \times \underline{E}^{(1)} = -c\, \underline{\omega} \times \underline{B}^{(1)} \quad -(43)$$

Using:

$$\frac{\omega_0}{\kappa} = 1 \quad -(44)$$

we find that:

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{B}^{(1)} &= \kappa \underline{B}^{(1)} = \frac{1}{c} \underline{\omega} \times \underline{E}^{(1)} \\ \underline{\nabla} \times \underline{E}^{(1)} &= \kappa \underline{E}^{(1)} = -c\, \underline{\omega} \times \underline{B}^{(1)} \end{aligned}} \quad -(45)$$

<u>These are extended Beltrami equations.</u>
They can be written as:

$$\underline{B}^{(1)} = \frac{1}{\omega} \underline{\omega} \times \underline{E}^{(1)} \quad -(46)$$

$$\underline{E}^{(1)} = -\frac{c^2}{\omega} \underline{\omega} \times \underline{B}^{(1)} \quad -(47)$$

using

$$\omega = c\kappa \quad -(48)$$

) In this notation:

$$\underline{\omega} = \kappa \underline{R} = \underline{\kappa} \quad -(49)$$

$$\omega = c\kappa = c\omega_0 \quad -(50)$$

So:

$$\underline{B} = \frac{1}{c\kappa}\, \underline{\kappa} \times \underline{E} \quad -(51)$$

$$\underline{E} = -\frac{c}{\kappa}\, \underline{\kappa} \times \underline{B} \quad -(52)$$

or

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{B} &= \frac{1}{c}\, \underline{\kappa} \times \underline{E} = \kappa \underline{B} \\ \underline{\nabla} \times \underline{E} &= -c\, \underline{\kappa} \times \underline{B} = \kappa \underline{E} \end{aligned}} \quad -(53)$$

For the general $\underline{\kappa}$, or spin connection vector:

$$\underline{\omega} = \underline{\kappa} \quad -(54)$$

Eqs. (53) can be solved numerically. The complete set of equations is:

$$\underline{\omega} \cdot \underline{B} = 0 \quad -(55)$$

$$\underline{\omega} \cdot \underline{E} = 0 \quad -(56)$$

$$\underline{\nabla} \times \underline{B} = \omega \underline{B} = \frac{1}{c}\, \underline{\omega} \times \underline{E} \quad -(57)$$

$$\underline{\nabla} \times \underline{E} = \omega \underline{E} = -c\, \underline{\omega} \times \underline{B} \quad -(58)$$

where

$$\omega^\mu = (\omega, \underline{\omega}) \quad -(59)$$

8) There are four equations in four unknowns, ω, $\underline{\omega}$, $\underline{B}$ and $\underline{E}$. These equations produce force free helicoidal flow and should produce transverse and longitudinal components, including $\underline{B}^{(3)}$ in the vacuum.