257(8): Rigorous Derivation of the Beltrami Equation from Cartan Geometry

In this derivation use the Cartan identity:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b := \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_c - \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b \quad -(1)$$

In the absence of a magnetic monopole:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = \underline{0} \quad -(2)$$

so:

$$\underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b = \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b \quad -(3)$$

Assume that the spin connection has the property:

$$\underline{\omega}^a{}_b = \epsilon^a{}_{bc}\, \underline{\omega}^c \quad -(4)$$

in space, where $\epsilon^a{}_{bc}$ is the 3-D totally antisymmetric unit tensor, then eq. (3) reduces to:

$$\underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^c = \underline{\omega}^c \cdot \underline{\nabla} \times \underline{q}^b \quad -(5)$$

An example of this is:

$$\underline{A}^{(2)} \cdot \underline{\nabla} \times \underline{\omega}^{(1)} = \underline{\omega}^{(1)} \cdot \underline{\nabla} \times \underline{A}^{(2)} \quad -(6)$$

in the complex circular basis ((1), (2), (3)). Here we have used the ECE hypothesis:

$$\underline{A} = A^{(0)}\, \underline{q} \quad -(7)$$

The absence of a magnetic monopole means that:

$$\underline{\omega}^a{}_b \cdot \underline{B}^b = \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad -(8)$$

where:

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(9)$$

Using eq. (4):

$$\underline{R}^c(\text{spin}) = \underline{\nabla} \times \underline{\omega}^c - \underline{\omega}^b \times \underline{\omega}^a \quad -(10)$$

The complex circular basis is defined by:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i\,\underline{e}^{(3)*} \quad -(11)$$

$$\underline{e}^{(3)} \times \underline{e}^{(1)} = i\,\underline{e}^{(2)*} \quad -(12)$$

$$\underline{e}^{(2)} \times \underline{e}^{(3)} = i\,\underline{e}^{(1)*} \quad -(13)$$

so:

$$\underline{e}^{(1)*} = \underline{e}^{(2)} = -i\,\underline{e}^{(2)} \times \underline{e}^{(3)} \quad -(14)$$

$$\underline{e}^{(2)*} = \underline{e}^{(1)} = -i\,\underline{e}^{(3)} \times \underline{e}^{(1)} \quad -(15)$$

$$\underline{e}^{(3)*} = \underline{e}^{(3)} = -i\,\underline{e}^{(1)} \times \underline{e}^{(2)} \quad -(16)$$

It follows that:

$$\underline{R}^{(1)}(\text{spin}) = \underline{\nabla} \times \underline{\omega}^{(1)} + i\,\underline{\omega}^{(3)} \times \underline{\omega}^{(1)}$$

$$\underline{R}^{(2)}(\text{spin}) = \underline{\nabla} \times \underline{\omega}^{(2)} + i\,\underline{\omega}^{(2)} \times \underline{\omega}^{(3)}$$

$$\underline{R}^{(3)}(\text{spin}) = \underline{\nabla} \times \underline{\omega}^{(3)} + i\,\underline{\omega}^{(1)} \times \underline{\omega}^{(2)}$$

$$-(17)$$

3) Similarly:

$$\underline{B}^{(1)} = \underline{\nabla} \times \underline{A}^{(1)} + i\underline{\omega}^{(3)} \times \underline{A}^{(1)}$$

$$\underline{B}^{(2)} = \underline{\nabla} \times \underline{A}^{(2)} + i\underline{\omega}^{(2)} \times \underline{A}^{(3)} \qquad -(18)$$

$$\underline{B}^{(3)} = \underline{\nabla} \times \underline{A}^{(3)} + i\underline{\omega}^{(1)} \times \underline{A}^{(2)}$$

Eq. (8) can be written as:

$$\underline{\omega}^{(1)} \cdot \underline{B}^{(2)} = \underline{A}^{(1)} \cdot \underline{R}^{(2)}(\text{Spin}) \qquad -(19)$$

i.e.

$$\underline{\omega}^{(1)} \cdot \left(\underline{\nabla} \times \underline{A}^{(2)} + i\underline{\omega}^{(2)} \times \underline{A}^{(3)}\right) \qquad -(20)$$

$$= \underline{A}^{(1)} \cdot \left(\underline{\nabla} \times \underline{\omega}^{(2)} + i\underline{\omega}^{(2)} \times \underline{\omega}^{(3)}\right)$$

from which:

$$\boxed{\underline{\omega}^{(i)} = \pm \frac{\kappa}{A^{(0)}} \underline{A}^{(i)}} \qquad -(21)$$

where $\qquad i = 1, 2, 3 \qquad -(22)$

In order to achieve self consistency with the original definition of $\underline{B}^{(3)}$ choose the minus sign in eq. (21). Then:

$$\underline{B}^{(3)} = \underline{\nabla} \times \underline{A}^{(3)} - i\frac{\kappa}{A^{(0)}} \underline{A}^{(1)} \times \underline{A}^{(2)} \qquad -(23)$$

4)

Also:

$$\underline{B}^{(1)} = \underline{\nabla} \times \underline{A}^{(1)} - i\frac{\kappa}{A^{(0)}}\underline{A}^{(3)} \times \underline{A}^{(1)} \quad -(24)$$

$$\underline{B}^{(2)} = \underline{\nabla} \times \underline{A}^{(2)} - i\frac{\kappa}{A^{(0)}}\underline{A}^{(2)} \times \underline{A}^{(3)} \quad -(25)$$

From eq. (2):

$$\underline{\nabla} \cdot \underline{\omega}^{(3)} \times \underline{A}^{(1)} = 0 \quad -(26)$$

and

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^{(1)} = 0 \quad -(27)$$

so

$$\underline{\nabla} \times \underline{A}^{(1)} = i\underline{\omega}^{(3)} \times \underline{A}^{(1)} \quad -(28)$$
$$= -i\frac{\kappa}{A^{(0)}}\underline{A}^{(3)} \times \underline{A}^{(1)}$$

and

$$\underline{\nabla} \times \underline{A}^{(2)} = i\underline{\omega}^{(2)} \times \underline{A}^{(3)} \quad -(29)$$
$$= -i\frac{\kappa}{A^{(0)}}\underline{A}^{(2)} \times \underline{A}^{(3)}$$

Multiply both sides of eqs. (11) to (13) by $A^{(0)}e^{i\phi}e^{-i\phi}$ where:

$$\phi = \omega t - \kappa Z \quad -(30)$$

This gives

$$\underline{A}^{(1)} \times \underline{A}^{(2)} = iA^{(0)}A^{(3)*}$$
$$\underline{A}^{(3)} \times \underline{A}^{(1)} = iA^{(0)}\underline{A}^{(2)*} \quad -(31)$$
$$\underline{A}^{(2)} \times \underline{A}^{(3)} = iA^{(0)}\underline{A}^{(1)*}$$

5) where:

$$\underline{A}^{(1)} = A^{(0)} \underline{e}^{(1)} e^{i\phi} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(32)$$

$$\underline{A}^{(2)} = A^{(0)} \underline{e}^{(2)} e^{-i\phi} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} \quad -(33)$$

$$\underline{A}^{(3)} = A^{(0)} \underline{e}^{(3)} = A^{(0)} \underline{k} \quad -(34)$$

From eqs. (28), (29) and (31):

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} = -\frac{i\kappa}{A^{(0)}} \underline{A}^{(3)} \times \underline{A}^{(1)} \quad -(35)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} = -\frac{i\kappa}{A^{(0)}} \underline{A}^{(2)} \times \underline{A}^{(3)} \quad -(36)$$

and these are Beltrami equations, QED. Here κ is constant and finite. The third equation is:

$$\underline{\nabla} \times \underline{A}^{(3)} = 0 \, \underline{A}^{(3)} \quad -(37)$$

In UFT 256 a particular case of these equations was used with:

$$\underline{\omega}^{(3)} = -\frac{1 \underline{k}}{A^{(0)}} \quad -(38)$$

$$-(39)$$

i.e.

$$\underline{\nabla} \times \underline{q}^{(1)} = -i \underline{k} \times \underline{q}^{(1)}$$

6) Note that in free space:

$$\left.\begin{aligned}\underline{\nabla}\cdot\underline{B}^{(1)} &= 0\\ \underline{\nabla}\times\underline{E}^{(1)}+\frac{\partial\underline{B}^{(1)}}{\partial t} &= \underline{0}\\ \underline{\nabla}\cdot\underline{E}^{(1)} &= 0\\ \underline{\nabla}\times\underline{B}^{(1)}-\frac{1}{c^2}\frac{\partial\underline{E}^{(1)}}{\partial t} &= \underline{0}\end{aligned}\right] \quad -(32)$$

and similarly for index (2). For index (3):

$$\underline{\nabla}\cdot\underline{B}^{(3)} = 0 \quad -(33)$$

$$\underline{\nabla}\times\underline{B}^{(3)} = \underline{0} \quad -(34)$$

Therefore:

$$\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}\left(i\underline{i}+\underline{j}\right)e^{i\phi} \quad -(35)$$

$$\underline{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}\left(-i\underline{i}+\underline{j}\right)e^{-i\phi} \quad -(36)$$

$$\underline{B}^{(3)} = B^{(0)}\underline{k} \quad -(37)$$

$$\underline{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}\left(\underline{i}-i\underline{j}\right)e^{i\phi} \quad -(38)$$

$$\underline{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}}\left(\underline{i}+i\underline{j}\right)e^{-i\phi} \quad -(39)$$

$$\underline{E}^{(3)} = \underline{0} \quad -(40)$$

7) and

$$\underline{\nabla} \times \underline{B}^{(1)} = \kappa \underline{B}^{(1)} \quad -(41)$$

$$\underline{\nabla} \times \underline{B}^{(2)} = \kappa \underline{B}^{(2)} \quad -(42)$$

$$\underline{\nabla} \times \underline{B}^{(3)} = 0 \, \underline{B}^{(3)} \quad -(43)$$

$$\underline{\nabla} \times \underline{E}^{(1)} = \kappa \underline{E}^{(1)} \quad -(44)$$

$$\underline{\nabla} \times \underline{E}^{(2)} = \kappa \underline{E}^{(2)} \quad -(45)$$

These are Beltrami equations QED.

The B Cyclic Theorem is:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = i B^{(0)} \underline{B}^{(3)*} \quad -(46)$$

$$\underline{B}^{(2)} \times \underline{B}^{(3)} = i B^{(0)} \underline{B}^{(1)*} \quad -(47)$$

$$\underline{B}^{(3)} \times \underline{B}^{(1)} = i B^{(0)} \underline{B}^{(2)*} \quad -(48)$$

In free space:

$$\underline{E}^{(1)} \times \underline{E}^{(2)} = c^2 \underline{B}^{(1)} \times \underline{B}^{(2)} \quad -(49)$$

However:

$$\underline{E}^{(3)} = \underline{0} \quad -(50)$$

and there is no electric equivalent of the inverse Faraday effect. The $\underline{E}^{(3)}$ field is not observed experimentally, and it is a polar vector. The conjugate product must give an axial vector, or pseudovector. Therefore $\underline{\omega}^{(3)}$ and $\underline{A}^{(3)}$ are regarded as pseudovectors. The Cartesian frame for example is:

$$\underline{i} \times \underline{j} = \underline{k} \quad -(50)$$

$$\underline{k} \times \underline{i} = \underline{j} \quad -(51)$$

$$\underline{j} \times \underline{k} = \underline{i} \quad -(52)$$

in one sense of handedness or chirality, but in the opposite sense:

$$\underline{i} \times \underline{j} = -\underline{k} \quad -(53)$$

$$\underline{k} \times \underline{i} = -\underline{j} \quad -(54)$$

$$\underline{j} \times \underline{k} = -\underline{i} \quad -(55)$$

If $\underline{i}$ and $\underline{j}$ are regarded as polar unit vectors, then $\underline{k}$ is an axial unit vector. Eq. (4) means that the spin connection vector is dual to an antisymmetric tensor in the internal space a, b, c.
