257(3): Reduction of General Solution to Conjugate Plane Waves and $\underline{B}^{(3)}$.

One general solution of the Beltrami equation is:

$$B_X = f_2 \sin \partial f_Z + f_3 \cos \partial f_Y \quad -(1)$$

$$B_Y = f_3 \sin \partial f_X + f_1 \cos \partial f_Z \quad -(2)$$

$$B_Z = f_1 \sin \partial f_Y + f_2 \cos \partial f_X \quad -(3)$$

and the correct version of another solution has been given by Dr. Horst Eckardt from computer algebra:

$$(\underline{\nabla} \times \underline{B})_X = -\frac{\partial w}{\partial Y} = B_X \quad -(4)$$

$$(\underline{\nabla} \times \underline{B})_Y = \frac{\partial w}{\partial X} = B_Y \quad -(5)$$

$$(\underline{\nabla} \times \underline{B})_Z = w = -\left(\left(\frac{\partial \phi}{\partial X}\right)^2 + \left(\frac{\partial \phi}{\partial Y}\right)^2\right) = B_Z \quad -(6)$$

It follows that:

$$\frac{d^2\phi}{dY^2} + \frac{d^2\phi}{dX^2} = \frac{dB_Y}{dX} - \frac{dB_X}{dB_Y} = w \quad -(7)$$

which gives:

$$(\underline{\nabla} \times \underline{B})_Z = B_Z \quad -(8)$$

This is the Z component of:

$$\underline{\nabla} \times \underline{B} = \underline{B} \quad -(9)$$

2) This solution is therefore:

$$(\underline{\nabla} \times \underline{B})_X = B_X = \frac{dB_Z}{dY} - \frac{dB_Y}{dZ} \quad -(10)$$

$$(\underline{\nabla} \times \underline{B})_Y = B_Y = \frac{dB_Y}{dZ} - \frac{dB_Z}{dY} \quad -(11)$$

$$(\underline{\nabla} \times \underline{B})_Z = B_Z = \frac{dB_Y}{dX} - \frac{dB_X}{dY} \quad -(12)$$

More generally:

$$(\underline{\nabla} \times \underline{B})_X = \alpha_X B_X \quad -(13)$$

$$(\underline{\nabla} \times \underline{B})_Y = \alpha_Y B_Y \quad -(14)$$

$$(\underline{\nabla} \times \underline{B})_Z = \alpha_Z B_Z \quad -(15)$$

In the case of conjugate plane waves and $\underline{B}^{(3)}$ we have:

$$\underline{\nabla} \times \underline{B}^{(1)} = \kappa \underline{B}^{(1)} \quad -(16)$$

$$\underline{\nabla} \times \underline{B}^{(2)} = \kappa \underline{B}^{(2)} \quad -(17)$$

$$\underline{\nabla} \times \underline{B}^{(3)} = 0 \, \underline{B}^{(3)} \quad -(18)$$

and

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = i B^{(0)} \underline{B}^{(3)*} \quad -(19)$$

where:

$$\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) \exp(i(\omega t - \kappa Z)) \quad -(20)$$

$$\underline{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) \exp(-i(\omega t - \kappa Z)) \quad -(21)$$

$$\underline{B}^{(3)} = B^{(0)} \underline{k} \quad -(22)$$

3) For $\underline{B}^{(1)}$:

$$B_x^{(1)} = \frac{B^{(0)}}{\sqrt{2}} e^{i\phi}, \quad B_y^{(1)} = -\frac{iB^{(0)}}{\sqrt{2}} e^{i\phi} \quad -(23)$$

where

$$\phi = \omega t - \kappa Z \quad -(24)$$

Therefore:

$$\left(\underline{\nabla} \times \underline{B}^{(1)}\right)_x = \kappa B_x^{(1)} = -\frac{dB_y^{(1)}}{dZ} \quad -(25)$$

$$\left(\underline{\nabla} \times \underline{B}^{(1)}\right)_y = \kappa B_y^{(1)} = \frac{dB_x^{(1)}}{dZ} \quad -(26)$$

So

$$\kappa B_x^{(1)} = -\frac{dB_y^{(1)}}{dZ} \quad -(27)$$

$$\kappa B_y^{(1)} = \frac{dB_x^{(1)}}{dZ} \quad -(28)$$

For this plane wave:

$$\frac{dB_y^{(1)}}{dX} = \frac{dB_x^{(1)}}{dY} = 0 \quad -(29)$$

so

$$\underline{\nabla} \times \underline{B}_Z^{(1)} = 0 B_Z^{(1)} \quad -(30)$$

with

$$B_Z^{(1)} = 0 \quad -(31)$$

From eqs. (27) and (28):

$$\frac{d^2 B_x^{(1)}}{dZ^2} + \frac{d^2 B_y^{(1)}}{dZ^2} = \kappa\left(\frac{dB_y^{(1)}}{dZ} - \frac{dB_x^{(1)}}{dZ}\right) \quad -(32)$$

4)

So

$$\frac{d^2 B_X^{(1)}}{dZ^2} + \frac{d^2 B_Y^{(1)}}{dZ^2} = -\kappa^2 \left(B_X^{(1)} + B_Y^{(1)} \right) \quad -(33)$$

and

$$\frac{d^2 B_X^{(1)}}{dZ^2} = -\kappa^2 B_X^{(1)} \quad -(34)$$

$$\frac{d^2 B_Y^{(1)}}{dZ^2} = -\kappa^2 B_Y^{(1)} \quad -(35)$$

These structures are part of the Proca type wave equation:

$$(\Box + \kappa^2) A_\mu^a = 0 \quad -(36)$$

which emerges from the ECE wave equation:

$$(\Box + \kappa^2) q_\mu^a = 0 \quad -(37)$$

From eq. (16)

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}^{(1)}) = \kappa \, \underline{\nabla} \times \underline{B}^{(1)} \quad -(38)$$

$$= \kappa^2 \underline{B}^{(1)}$$

this is a Trkalian type of construction, in which:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}^{(1)}) = \underline{\nabla} (\underline{\nabla} \cdot \underline{B}^{(1)}) - \nabla^2 \underline{B}^{(1)} \quad -(39)$$

However:

$$\underline{\nabla} \cdot \underline{B}^{(1)} = 0 \quad -(40)$$

) so

$$-\nabla^2 \underline{B}^{(1)} = \kappa^2 \underline{B}^{(1)} \quad -(41)$$

which combines eqs. (34) and (35), QED.

The covariantly relativistic format of eq. (41) is

$$\Box \underline{B}^{(1)} = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \underline{B}^{(1)} \quad -(42)$$

where $\Box$ is the d'Alembertian. The $B^{(3)}$ field implies finite photon mass, so the relation between ω and κ in eq. (20) is given by:

$$E^2 = p^2 c^2 + m^2 c^4 \quad -(43)$$

where m is the photon mass, i.e.

$$\omega^2 = c^2 \kappa^2 + \left(\frac{mc^2}{\hbar} \right)^2 \quad -(44)$$

Using the quantum postulates:

$$E = i\hbar \frac{\partial}{\partial t}, \quad \underline{p} = -i\hbar \underline{\nabla} \quad -(45)$$

produces:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \phi = 0 \quad -(46)$$

i.e.

$$\boxed{\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \underline{B}^{(1)} = \underline{0}} \quad -(47)$$

which is a Proca equation.

6) Note that:

$$\Box \underline{B}^{(1)} = \left(k^2 - \frac{\omega^2}{c^2} \right) \underline{B}^{(1)} \quad - (48)$$

$$= -\left(\frac{mc}{\hbar} \right)^2 \underline{B}^{(1)}$$

so

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \underline{B}^{(1)} = 0. \quad - (49)$$

The d'Alembertian is:

$$\Box \underline{B}^{(1)} = \frac{1}{c^2} \frac{\partial^2 \underline{B}^{(1)}}{\partial t^2} + \underline{\nabla} \times \left(\underline{\nabla} \times \underline{B}^{(1)} \right) \quad - (50)$$

so

$$\underline{\nabla} \times \left(\underline{\nabla} \times \underline{B}^{(1)} \right) = \left(\Box - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \underline{B}^{(1)} \quad - (51).$$

$$= k^2 \underline{B}^{(1)}$$

$$= \frac{1}{c^2} \left(\omega^2 - \omega_0^2 \right) \underline{B}^{(1)}$$

Here

$$\omega_0 = mc^2 / \hbar \quad - (52)$$

which is the rest frequency of the photon.

In order to reduce eqns (1) to (3) to a plane wave note that if:

$$\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \left(\cos\phi + i \sin\phi \right) \quad - (53)$$

where

$$e^{i\phi} = \cos\phi + i\sin\phi \quad - (54)$$

7) Then the real parts of $B_X^{(1)}$, $B_Y^{(1)}$ and $B_Z^{(1)}$ are:

$$\mathrm{Re}\, B_X^{(1)} = \frac{B^{(0)}}{\sqrt{2}} \cos\phi \quad - (55)$$

$$\mathrm{Re}\, B_Y^{(1)} = \frac{B^{(0)}}{\sqrt{2}} \sin\phi \quad - (56)$$

$$\mathrm{Re}\, B_Z^{(1)} = 0 \quad - (57)$$

From eqs. (1) to (3) and (55) to (57):

$$f_3 \cos 2f_Y = \frac{B^{(0)}}{\sqrt{2}} \cos\phi \quad - (58)$$

$$f_3 \sin 2f_X = \frac{B^{(0)}}{\sqrt{2}} \sin\phi \quad - (59)$$

So:

$$2f_Y = 2f_X = \phi = \omega t - \kappa Z \quad - (60)$$

$$f_3 = \frac{B^{(0)}}{\sqrt{2}}, \quad - (61)$$

$$f_1 = f_2 = 0, \quad - (62)$$

QED - The real parts of $\underline{B}^{(1)}$ and $\underline{B}^{(2)}$ are the same, so the results for $\underline{B}^{(2)}$ are the same.

For the longitudinal $\underline{B}^{(3)}$ field:

$$\underline{B}^{(3)} = B^{(0)} \underline{k} \quad - (63)$$

$$\underline{\nabla} \times \underline{B}^{(3)} = 0\, \underline{B}^{(3)} \quad - (64)$$

Here:

$$B_Z = B^{(0)} \quad -(65)$$

Comparing eqns. (3) and (65):

$$B_Z = f_1 \sin \partial f_Y + f_2 \cos \partial f_X \quad -(66)$$

$$= \text{constant}$$

Therefore

$$\partial f_X = 0 = \partial f_Y \quad -(67)$$

is the simplest solution. So:

$$B_Z = f_2 \quad -(68)$$

So it is possible to express transverse plane wave and $\underline{B}^{(3)}$ in terms of eqns (1) to (3). The transverse plane wave are:

$$\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} = f_3 (\underline{i} - i\underline{j}) e^{i f_X / 2} \quad -(69)$$

$$\underline{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} = f_3 (\underline{i} + i\underline{j}) e^{-i f_X / 2} \quad -(70)$$

with:

$$f_X = f_Y. \quad -(71)$$

So:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = \underline{B} \times \underline{B}^* = 2i f_3^2 \quad -(70)$$

i.e.

$$\underline{B}^{(3)} = B^{(0)} \underline{k} \quad -(71)$$

9) We have:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = iB^{(0)}\underline{B}^{(3)} = iB^{(0)2}\underline{k}$$

$$= f_3^2 \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -i & 0 \\ 1 & i & 0 \end{vmatrix} = 2if_3^2\underline{k} \quad -(72)$$

so

$$f_3 = \frac{B^{(0)}}{\sqrt{2}} \quad -(73)$$

Q.E.D.

$$\underline{B}^{(3)} = \sqrt{2}f_3\underline{k} \quad -(74)$$

Result

The B Cyclic Theorem:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = iB^{(0)}\underline{B}^{(3)*} \quad -(74)$$

et cyclicum

is obtained from eqs. (1) to (3) with:

$$f_1 = f_2 = 0 \quad -(75)$$

$$f_3 = B^{(0)}/\sqrt{2} \quad -(76)$$

$$\phi = \omega t - \kappa Z = \frac{fx}{2} = \frac{fy}{2} \quad -(77)$$

with

$$\underline{\nabla} \times \underline{B}^{(1)} = \kappa\underline{B}^{(1)} \quad -(78)$$

$$\underline{\nabla} \times \underline{B}^{(2)} = \kappa\underline{B}^{(2)} \quad -(79)$$

$$\underline{\nabla} \times \underline{B}^{(3)} = 0\,\underline{B}^{(3)} \quad -(80)$$

Conversely, eqs. (1) and (2) become a B cyclic theorem if:

$$\underline{B}^{(1)} = f_3 (\underline{i} - i \underline{j}) \exp(i f_x / 2) \quad - (81)$$

$$\underline{B}^{(2)} = f_3 (\underline{i} + i \underline{j}) \exp(-i f_x / 2) \quad - (82)$$

$$\underline{B}^{(3)} = \sqrt{2} f_3 \underline{k}, \quad - (83)$$

$$f_x = f_y \quad - (84)$$

$$f_1 = f_2 = 0 \quad - (85)$$

$$B_x = B_x^{(1)} = f_3 \cos 2 f_x \quad - (86)$$

$$B_y = B_y^{(1)} = f_3 \sin 2 f_x \quad - (87)$$

$$\underline{\nabla} \times \underline{B} = \alpha \underline{B} \quad - (88)$$

where α is a constant. We have:

$$\underline{B} = B_x \underline{i} + B_y \underline{j} \quad - (89)$$

so

$$\underline{\nabla} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ B_x & B_y & 0 \end{vmatrix} \quad - (90)$$

$$= - \underline{i} \frac{\partial B_y}{\partial Z} + \underline{j} \frac{\partial B_x}{\partial Z}$$

Here:

$$B_Y = f_3 \sin \alpha f_X = f_3 \sin(\omega t - \kappa Z) \quad -(91)$$

$$B_X = f_3 \cos \alpha f_X = f_3 \cos(\omega t - \kappa Z) \quad -(92)$$

So

$$\frac{\partial B_Y}{\partial Z} = -\kappa f_3 \cos \phi = -\kappa f_3 \cos \alpha f_X \quad -(93)$$

$$\frac{\partial B_X}{\partial Z} = \kappa f_3 \sin \phi = \kappa f_3 \sin \alpha f_X \quad -(93a)$$

Therefore

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad -(94)$$

i.e.

$$\underline{\nabla} \times \underline{B}^{(1)} = \kappa \underline{B}^{(1)} \quad -(95)$$

QED

This analysis shows that eqs. (1) to (3) can give a generalization of the B Cyclic Theorem, which will be the subject of the next note.

It also shows that the $\underline{B}^{(3)}$ field can be generalized to:

$$\boxed{\begin{aligned} \underline{B}^{(3)} &= B_Z \underline{k} \\ &= (f_1 \sin \alpha f_Y + f_2 \cos \alpha f_X) \underline{k} \end{aligned}} \quad -(96)$$

If in the vacuum:

$$\underline{\nabla} \times \underline{B} = \alpha \underline{B} \quad -(97)$$

12) then:

$$\underline{B} = (f_2 \sin \alpha f_z + f_3 \cos \alpha f_y)\,\underline{i} + (f_3 \sin \alpha f_x + f_1 \cos \alpha f_z)\,\underline{j} + (f_1 \sin \alpha f_y + f_2 \cos \alpha f_x)\,\underline{k} \quad -(98)$$

If: $$B_Z = \text{constant} \quad -(99)$$

then: $$f_x = f_y = 0 \quad -(100)$$

and: $$\underline{B} = f_2 \sin \alpha f_Z\,\underline{i} + f_1 \cos \alpha f_Z\,\underline{j} + f_2\,\underline{k} \quad -(101)$$

Eq. (101) gives two conjugate plane waves and a $\underline{B}^{(3)}$ field, which in Beltrami flow is a field component along Z.

Graphics and Animation

Eq. (98) and Eq. (101) can be graphed and animated. They are SOLF solutions of the Beltrami equation:

$$\underline{\nabla} \times \underline{B} = \alpha\,\underline{B} \quad -(102)$$