157(7). Spin Connection and Vector Potential, General Definition of the Beltrami Condition.

It has been shown in previous work that the space part of the Cartan identity is:

$$\underline{\nabla} \cdot \underline{\omega} \times \underline{q} := \underline{q} \cdot \underline{\nabla} \times \underline{\omega} - \underline{\omega} \cdot \underline{\nabla} \times \underline{q} \quad -(1)$$

which is a well known vector identity. The condition

$$\underline{\nabla} \cdot \underline{\omega} \times \underline{q} = 0 \quad -(2)$$

means that:

$$\underline{\omega} \cdot \underline{\nabla} \times \underline{q} = \underline{q} \cdot \underline{\nabla} \times \underline{\omega} \quad -(3)$$

The ECE hypothesis for each index a gives:

$$\underline{A} = A^{(0)} \underline{q} \quad -(4)$$

So

$$\underline{\nabla} \cdot \underline{\omega} \times \underline{A} = 0 \quad -(5)$$

and

$$\underline{\omega} \cdot \underline{\nabla} \times \underline{A} = \underline{A} \cdot \underline{\nabla} \times \underline{\omega} \quad -(6)$$

The magnetic field is:

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(7)$$

so from eqns. (5) and (7):

$$\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot \underline{\nabla} \times \underline{A} - \underline{\nabla} \cdot \underline{\omega} \times \underline{A}$$
$$= 0 \quad -(8)$$

Therefore eq. (6) implies eq. (8) and vice versa. The absence of a magnetic monopole

) in the simplified ECE theory means:

$$\underline{\omega} \cdot \underline{B} = \underline{A} \cdot \underline{\nabla} \times \underline{\omega} \quad -(9)$$

where

$$\underline{R}(\text{spin}) = \underline{\nabla} \times \underline{\omega} \quad -(10)$$

From eqs. (6) and (9):

$$\underline{\omega} \cdot \underline{B} = \underline{A} \cdot \underline{\nabla} \times \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \times \underline{A} \quad -(11)$$

so

$$\boxed{\underline{B} = \underline{\nabla} \times \underline{A}} \quad -(12)$$

and

$$\boxed{\underline{\omega} \times \underline{A} = \underline{0}} \quad -(13)$$

These results are consistent with:

$$p^\mu = eA^\mu = \hbar \kappa^\mu = \hbar \omega^\mu \quad -(14)$$

from the minimal prescription. So:

$$\boxed{\underline{\omega} = \frac{e}{\hbar} \underline{A}, \quad \omega_0 = \frac{e}{\hbar} A_0} \quad -(15)$$

The spin connection is the potential within $e/\hbar$.

This is a general advance in ECE theory.

The $\underline{B}^{(3)}$ field is defined as:

$$\underline{B}^{(3)*} = -\frac{e}{\hbar} \underline{A}^{(1)} \times \underline{A}^{(2)} \quad -(16)$$

where:

$$eA^{(0)} = \hbar \kappa, \quad -(17)$$

So:

$$\underline{B}^{(3)*} = -\frac{\hbar}{e} \underline{\omega}^{(1)} \times \underline{\omega}^{(2)} \quad -(18)$$

$$\boxed{\underline{B}^{(3)*} = -\frac{A^{(0)}}{\kappa} \underline{\omega}^{(1)} \times \underline{\omega}^{(2)}} \quad -(19)$$

in general relativity. Here:

$$\underline{\omega}^{(1)} = \frac{e}{\hbar} \underline{A}^{(1)} = \frac{eA^{(0)}}{\sqrt{2}\hbar} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)}$$

$$= \frac{\kappa}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa z)} \quad -(20)$$

Therefore the spin connections are conjugate plane waves

$$\underline{\omega}^{(1)} = \frac{\kappa}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)} \quad -(21)$$

$$\underline{\omega}^{(2)} = \frac{\kappa}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i(\omega t - \kappa Z)} \quad -(22)$$

From eqs. (21) and (22):

$$\underline{\nabla} \times \underline{\omega}^{(1)} = \kappa \underline{\omega}^{(1)} \quad -(23)$$

$$\underline{\nabla} \times \underline{\omega}^{(2)} = \kappa \underline{\omega}^{(2)} \quad -(24)$$

4) These are Beltrami equations.

From eq. (6):

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(25)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} \quad -(26)$$

and these are also Beltrami equations.

As shown in previous notes there are many interesting solutions to these Beltrami equations apart from plane waves, all of them contain longitudinal components in vacuo, so the photon mass is identically non-zero.

Using eq. (12) for each conjugate index:

$$\begin{aligned} \underline{\nabla} \times \underline{B} &= \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) \\ &= \kappa \underline{\nabla} \times \underline{A} \\ &= \kappa \underline{B} \end{aligned} \quad -(27)$$

so the Beltrami equation:

$$\boxed{\underline{\nabla} \times \underline{B} = \kappa \underline{B}} \quad -(28)$$

has been derived from Cartan geometry. It has been shown that the Beltrami equation is a general result and always exists.

5) Therefore for free space electromagnetism:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(29)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(30)$$

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad -(31)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(32)$$

Eq. (30) is the Faraday law of induction.
From eqs. (30) and (31):

$$\underline{\nabla} \times \underline{E} + \frac{1}{\kappa}\frac{\partial}{\partial t}\left(\underline{\nabla} \times \underline{B}\right) = 0 \quad -(33)$$

and it follows that:

$$\underline{\nabla} \times \underline{E} = \kappa \underline{E} \quad -(34)$$

Therefore vacuum electromagnetism, or free field electromagnetism, is described <u>completely</u> by three Beltrami equations:

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{A} &= \kappa \underline{A} \\ \underline{\nabla} \times \underline{B} &= \kappa \underline{B} \\ \underline{\nabla} \times \underline{E} &= \kappa \underline{E} \end{aligned}} \quad -(35)$$

<u>This</u> is <u>a result of</u> general <u>relativity</u>.

6) The electric field strength is defined in general by:

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} - c\omega_0 \underline{A} + \phi \underline{\omega} \quad -(36)$$

where

$$\phi = cA_0 \quad -(37)$$

From eq. (15) and (36)

$$\phi \underline{\omega} = c\omega_0 \underline{A} \quad -(38)$$

so:

$$\boxed{\begin{aligned} \underline{E} &= -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \quad &-(39) \\ \underline{B} &= \underline{\nabla} \times \underline{A} \quad &-(40) \end{aligned}}$$

This is the same as the structure derived by Heaviside, but it has been derived from generally covariant unified field theory and general relativity.

Field Matter Interaction

In this case the electric field strength $\underline{E}$ is replaced in general by the electric displacement $\underline{D}$, and the magnetic flux density $\underline{B}$ by the magnetic field strength $\underline{H}$. The equations governing field matter interaction

7) are the Coulomb law and Ampère Maxwell law:

$$\underline{\nabla} \cdot \underline{D} = \rho \quad -(41)$$

$$\underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \quad -(42)$$

where

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} \quad -(43)$$

$$\underline{H} = \frac{1}{\mu_0} \left(\underline{B} - \underline{M} \right) \quad -(44)$$

where $\underline{P}$ is the polarization and $\underline{M}$ is the magnetization. If the material matter is not polarizable or magnetizable then eqs. (41) and (42) become:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(45)$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \quad -(46)$$

together with

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(47)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(48)$$

These four equations must be solved simultaneously. In general, when there is field matter interaction, $\underline{E}$ is not a Beltrami field, because from eqs. (35) and (45), the charge density would vanish.

8) Also in field matter interaction the free space condition:

$$eA^{\mu} = \hbar\omega^{\mu} \quad - (49)$$

no longer holds in general, so:

$$\phi\underline{\omega} \neq c\omega_0\underline{A} \quad - (50)$$

in general, and:

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} - c\omega_0\underline{A} + \phi\underline{\omega} \quad - (51)$$

The condition (13) still holds in field matter interaction because it is the result of assuming no magnetic monopole. So in field matter interaction $\underline{\omega}$ is parallel to $\underline{A}$, meaning that:

$$\omega_X A_Y - \omega_Y A_X = 0 \quad - (52)$$

$$\omega_Z A_X - \omega_X A_Z = 0 \quad - (53)$$

$$\omega_Y A_Z - \omega_Z A_Y = 0 \quad - (54)$$

The spin connection vector is:

$$\omega^{\mu} = (\omega_0, \omega_X, \omega_Y, \omega_Z) \quad - (55)$$

and the potential four vector is:

$$A^{\mu} = (A_0, A_X, A_Y, A_Z) \quad - (56)$$

so

$$A_X = \left(\frac{A_Y}{\omega_Y}\right)\omega_X \quad - (57)$$

The X component of $\underline{E}$ in eq. (51) is:

$$E_X = c\,(A_0\,\omega_X - \omega_0 A_X) + \cdots \quad -(58)$$

and this is not zero in general. The Maxwell Heaviside structure of eqs. (39) and (40) appears only if it is assumed in free space that:

$$p^\mu = eA^\mu = \hbar\omega^\mu = \hbar\kappa^\mu. \quad -(59)$$

In field matter interaction:

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \quad -(60)$$

and

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(61)$$

so a Beltrami equation:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad -(62)$$

is still consistent with eq. (61). From eqs. (60) and (62):

$$\kappa \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \quad -(63)$$

so the current density is related to the magnetic ~~field~~ flux density by:

$$\underline{J} = \frac{\kappa}{\mu_0}\underline{B} - \frac{1}{\mu_0 c^2}\frac{\partial \underline{E}}{\partial t} \quad -(64)$$

In magnetostatics, or for a slowly varying electric field:

$$\boxed{\underline{B} = \frac{\mu_0}{\kappa} \underline{J}} \quad -(65)$$

and

$$\underline{\nabla} \times \underline{B} = \frac{\mu_0}{\kappa} \underline{\nabla} \times \underline{J} = \kappa \underline{B} \quad -(66)$$

so

$$\boxed{\underline{B} = \frac{\mu_0}{\kappa^2} \underline{\nabla} \times \underline{J}} \quad -(67)$$

As discussed by Reed and Zaghloul and Barajas (Am. J. Phys., 58, 783 ff., (1990)) the Beltrami field $\underline{B}$ has longitudinal solutions and transverse solutions. From eq. (67) the longitudinal component gives rise to a circling current density. Beltrami fields also give rise to $\underline{B}^{(3)}$ solutions, in which case:

$$\boxed{\underline{B}^{(3)} = \frac{\mu_0}{\kappa} \underline{J}^{(3)}} \quad -(68)$$

and this may be the jet energies from the plane of a whirlpool galaxy.