256(6): Double Checking the Derivation of the Spin Connection Four Vector of a Vacuum Plane Wave.

The derivation starts from the equation:

$$\underline{\nabla} \times \underline{q}^a = -i\, \underline{\omega}^a{}_b \times \underline{q}^b \quad -(1)$$

which can be simplified to:

$$\underline{\nabla} \times \underline{q} = -i\, \underline{\omega} \times \underline{q} \quad -(2)$$

where

$$\underline{q} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j})\, e^{i(\omega t - \kappa Z)} \quad -(3)$$

is the vacuum plane wave. Here:

$$\underline{\nabla} \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ e^{i\phi} & -ie^{i\phi} & 0 \end{vmatrix} \quad -(4)$$

where

$$\phi = \omega t - \kappa Z \quad -(5)$$

So

$$\underline{\nabla} \times \underline{q} = \frac{\partial}{\partial Z} i e^{i\phi}\, \underline{i} + \frac{\partial}{\partial Z} e^{i\phi}\, \underline{j}$$

$$= \frac{\kappa}{\sqrt{2}} (\underline{i} - i\underline{j})\, e^{i\phi}$$

$$= \kappa\, \underline{q} \quad -(5)$$

QED

2) Therefore:

$$\underline{\nabla} \times \underline{q} = \kappa \underline{q} = -i\underline{\omega} \times \underline{q} \quad -(6)$$

$$= -\frac{i}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega_Z \\ 1 & -i & 0 \end{vmatrix} e^{i\phi}$$

$$= -\frac{i}{\sqrt{2}} \omega_Z \left(i\underline{i} + \underline{j} \right) e^{i\phi}$$

$$= \omega_Z \underline{q}$$

Therefore:

$$\boxed{\omega_Z = \kappa} \quad -(7)$$

QED The spin connection vector is therefore:

$$\underline{\omega} = \omega_Z \underline{k} \quad -(8)$$

As in note 256(2) the orbital curvature reduces to:

$$\underline{R}(\text{orb}) = -\underline{\nabla}\omega_0 - \frac{1}{c}\frac{\partial \underline{\omega}}{\partial t} \quad -(9)$$

and the spin curvature to:

$$\underline{R}(\text{spin}) = \underline{\nabla} \times \underline{\omega} \quad -(10)$$

where

$$\omega^\mu = (\omega_0, \underline{\omega}) \quad -(11)$$

3)

Now assume that ω_0 is a constant, then both the orbital and spin curvatures of a vacuum plane wave are zero.

The charge density of the vacuum plane wave is zero:

$$\rho = \epsilon_0 \, \underline{\omega} \cdot \underline{E} = 0 \quad -(12)$$

and this is a self consistent result, QED.

The current density of the vacuum plane wave is:

$$\underline{J} = \frac{\omega_0}{c} \underline{E} + \underline{\omega} \times \underline{B} \quad -(13)$$

Here:

$$\underline{\omega} \times \underline{B} = \frac{e^{i\phi}}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \kappa \\ i & 1 & 0 \end{vmatrix} B^{(0)}$$

$$= \frac{e^{i\phi} B^{(0)}}{\sqrt{2}} \left(-\kappa \, \underline{i} + \kappa i \, \underline{j} \right)$$

$$= -\frac{\kappa B^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) e^{i\phi}$$

$$= -\frac{\kappa}{c} \underline{E} \quad -(14)$$

because

$$E^{(0)} = c B^{(0)} \quad -(15)$$

Therefore:

4)

$$\underline{J} = \underline{0} \quad -(16)$$

if

$$\omega_0 = \kappa \quad -(17)$$

QED. The vacuum current density vanishes if eq. (17) is true. This is a self consistent result because the vacuum current density of a plane wave in the vacuum is zero. It is also consistent with the assumption that ω_0 is a constant, QED.

Therefore:

$$\boxed{\omega^\mu = \kappa^\mu} \quad -(18)$$

i.e.

$$(\omega_0, \underline{\omega}) = \left(\frac{\omega}{c}, \underline{\kappa}\right) \quad -(19)$$

QED. The spin connection four-vector of the vacuum plane wave is the wave four vector.

The absence of a magnetic monopole density means that:

$$\underline{\omega} \cdot \underline{B} = 0 \quad -(20)$$

and again this is self-consistent QED.

The absence of magnetic current density

) means that:

$$\underline{\omega} \times \underline{E} - c\omega_0 \underline{B} = \underline{0} \quad -(21)$$

Here $\underline{\omega} \times \underline{E} = \frac{e^{i\phi}}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \kappa \\ 1 & -i & 0 \end{vmatrix} E^{(0)}$

$$= \frac{\kappa E^{(0)}}{\sqrt{2}} e^{i\phi} \left(i\underline{i} + \underline{j} \right)$$

$$= c\kappa \underline{B} \quad -(22)$$

So again: $\omega_0 = \kappa \quad -(23)$

self consistently, QED.

The four vacuum equations are:

$$\boxed{\omega \wedge F = \omega_\mu \tilde{F}^{\mu\nu} = 0 \rightarrow \begin{array}{l} \underline{\omega} \cdot \underline{B} = 0 \\ \omega_0 \underline{B} - \frac{1}{c} \underline{\omega} \times \underline{E} = \underline{0} \end{array}}$$

$-(24)$

and:

$$\boxed{\omega \wedge \tilde{F} = \omega_\mu F^{\mu\nu} = 0 \longrightarrow \begin{array}{l} \underline{\omega} \cdot \underline{E} = 0 \\ \omega_0 \underline{E} + c\, \underline{\omega} \times \underline{B} = \underline{0} \end{array}}$$

$-(25)$