1)

254(8) : Hodge Dual and Duality Transform.

As in previous notes the Hodge dual and duality transform are clearly inter-related. On the simplest level the Hodge duals are given by:

$$\tilde{F}^{03} = F_{12} \quad -(1)$$

$$\tilde{F}^{01} = F_{23} \quad -(2)$$

$$\tilde{F}^{02} = F_{31} \quad -(3)$$

which suggests:

$$\partial_1 A_2 - \partial_2 A_1 = c\left(\partial^0 A^3 - \partial^3 A^0\right) \quad -(4)$$

$$\partial_3 A_1 - \partial_1 A_3 = c\left(\partial^0 A^2 - \partial^2 A^0\right) \quad -(5)$$

$$\partial_2 A_3 - \partial_3 A_2 = c\left(\partial^0 A^1 - \partial^1 A^0\right) \quad -(6)$$

The factor c is present because:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right) \quad -(7)$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad -(8)$$

$$A^\mu = \left(A^0, \underline{A}\right) = \left(\frac{\phi}{c}, \underline{A}\right) \quad -(9)$$

$$A_\mu = \left(A^0, -\underline{A}\right) = \left(\frac{\phi}{c}, -\underline{A}\right) \quad -(10)$$

Now consider for simplicity of argument only the free space Maxwell Heaviside equations:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(11)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \quad -(12)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(13)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0} \quad -(14)$$

It is well known that these equations remain unchanged under the duality transformation:

$$\underline{E} = ic\underline{B} \; ; \; \underline{B} = \frac{1}{ic} \underline{E} \quad -(15)$$

For example, eq. (14) transforms to:

$$\frac{1}{ic} \underline{\nabla} \times \underline{E} - \frac{1}{c^2} ic \frac{\partial \underline{B}}{\partial t} = 0 \quad -(16)$$

which is eq. (12), QED.

Therefore: $\quad -(17)$

$$\underline{E} \rightarrow ic\underline{B} = ic\underline{\nabla} \times \underline{A} = c\underline{\nabla} \times (i\underline{A})$$

$$\underline{B} \rightarrow \frac{1}{ic} \underline{E} = \frac{1}{ic} \left(-\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \right) \quad -(18)$$

$$= \underline{\nabla}(i\phi) + \frac{\partial (i\underline{A})}{\partial t}$$

Eqs. (17) and (18) are examples of eqns. (4) to (6). It can be seen that the four potential is complex valued. Eq. (17) is very useful to deduce the vector format of the Cartan identity as in the previous note.

3) In ECE theory:

$$\underline{E}^a \rightarrow c\left(\underline{\nabla} \times (i\underline{A})^a - \omega^a{}_b \times (i\underline{A}^b)\right) \quad -(19)$$

$$\underline{B}^a \rightarrow \frac{1}{ic}\left(-\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} - \omega^a{}_{0b}\underline{A}^b + \underline{\omega}^a{}_b\phi^b\right) \quad -(20)$$

In the presence of matter eqs. (11) and (12) remain the same but eqs. (13) and (14) become:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(21)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} \quad -(22)$$

so the duality transform (15) is augmented by:

$$\underline{E} \rightarrow ic\underline{B}, \quad \rho \rightarrow 0, \quad \underline{J} \rightarrow 0 \quad -(23)$$

$$\underline{B} \rightarrow \frac{1}{ic}\underline{E}, \quad 0 \rightarrow \rho, \quad \underline{0} \rightarrow \underline{J} \quad -(24)$$

In ECE theory eqs. (11) and (12) become:

$$D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^a{}_\mu{}^{\mu\nu} \quad -(25)$$

Eqs. (13) and (14) become:

$$D_\mu T^{a\mu\nu} = R^a{}_\mu{}^{\mu\nu} \quad -(26)$$

Eqs. (25) and (26) are respectively:

4)

$$\partial_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a\ \ \mu\nu}_{\ \mu} - \omega^a_{\ \mu b} \tilde{T}^{b\mu\nu} = 0 \quad -(27)$$

$$\partial_\mu T^{a\mu\nu} = R^{a\ \ \mu\nu}_{\ \mu} - \omega^a_{\ \mu b} T^{b\mu\nu} = j^{a\nu} \quad -(28)$$

The duality transform become:

$$\tilde{T}^{a\mu\nu} \rightleftarrows T^{a\mu\nu} ; \quad 0 \rightleftarrows j^{a\nu} \quad -(29)$$

Eq. (25) is the same as the antisymmetrized tensor product:

$$D_\mu T^a_{\ \nu\rho} + D_\rho T^a_{\ \mu\nu} + D_\nu T^a_{\ \rho\mu} = R^a_{\ \mu\nu\rho} + R^a_{\ \rho\mu\nu} + R^a_{\ \nu\rho\mu} \quad -(30)$$

and eq. (26) is the same as the antisymmetrized tensor product:

$$D_\mu \tilde{T}^a_{\ \nu\rho} + D_\rho \tilde{T}^a_{\ \mu\nu} + D_\nu \tilde{T}^a_{\ \rho\mu} = \tilde{R}^a_{\ \mu\nu\rho} + \tilde{R}^a_{\ \rho\mu\nu} + \tilde{R}^a_{\ \nu\rho\mu} \quad -(31)$$

In differential form notation eqs (25) or (30) are both equivalent to the Cartan identity:

$$D \wedge T^a := q^b \wedge R^a_{\ b} \quad -(32)$$

and eqs (26) or (31) are equivalent to the Evans identity:

$$D \wedge \tilde{T}^a := q^b \wedge \tilde{R}^a_{\ b} \quad -(33)$$

Eq. (32) gives the homogeneous field equation:

5)
$$d \wedge T^a = q^b \wedge R^a{}_b - \omega^a{}_b \wedge T^b = 0 \quad -(34)$$

and eq. (33) give the inhomogeneous field equation:

$$d \wedge \tilde{T}^a = q^b \wedge \tilde{R}^a{}_b - \omega^a{}_b \wedge \tilde{T}^b \neq 0$$
$$= j^a \quad -(35)$$

Eqs. (11) and (12) of the Maxwell Heaviside theory are:

$$d \wedge F = 0 \quad -(36)$$

and Eqs. (13) and (14) are:

$$d \wedge \tilde{F} = \mu_0 j \quad -(37)$$

As shown in previous notes eq. (30) reduces to:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b - \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(38)$$

and eq. (37) reduces to:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = 0 \quad -(39)$$

In view of eq. (17) for free fields, eq. (39) implies:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{B}^c &= 0 \quad -(40) \\ \underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{E}^c &= 0 \quad -(41) \end{aligned}}$$