254(7): Cartan Identities in Vector Format

The structure equations are:

$$\underline{T}^a(\text{spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \quad -(1)$$

$$\underline{T}^a(\text{orbital}) = \left(\underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b\right)_{HD} \quad -(2)$$

and

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(3)$$

$$\underline{R}^a{}_b(\text{orbital}) = \left(\underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b\right)_{HD} \quad -(4)$$

where HD denotes "Hodge dual".

The Cartan identity is, for spin torsion:

$$\underline{\nabla} \cdot \underline{T}^a(\text{spin}) + \underline{\omega}^a{}_b \times \underline{T}^b(\text{spin}) \quad -(5)$$
$$:= \underline{q}^b \cdot \left(\underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_b \times \underline{\omega}^b{}_c\right)$$

i.e.

$$\underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b\right) \quad -(6)$$
$$+ \underline{\omega}^a{}_b \cdot \left(\underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b\right)$$
$$= \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{q}^b \cdot \underline{\omega}^a{}_b \times \underline{\omega}^b{}_c$$

Now use:

$$\underline{q}^b \cdot \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b = \underline{q}^c \cdot \underline{\omega}^a{}_b \times \underline{\omega}^b{}_c \quad -(7)$$
$$= \underline{\omega}^a{}_b \cdot \underline{\omega}^b{}_c \times \underline{q}^c$$

2) to find that:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c := \underline{\omega}^a{}_b \cdot \underline{\nabla} \times q^b - \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(8)$$

which is the vector identity: $-(9)$

$$\underline{\nabla} \cdot \underline{F} \times \underline{G} = \underline{G} \cdot \underline{\nabla} \times \underline{F} - \underline{F} \cdot \underline{\nabla} \times \underline{G}$$

therefore eq. (8) is an identity, QED.

Now assume the structure of the homogeneous field equation:

$$\underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{q}^b - \underline{\omega}^b{}_c \times \underline{q}^c \right) = 0 \quad -(10)$$

Then it follows that

$$\boxed{\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = 0} \quad -(11)$$

From eqs. (6) and (10):

$$\underline{\omega}^a{}_b \cdot \left(\underline{\nabla} \times \underline{q}^b - \underline{\omega}^b{}_c \times \underline{q}^c \right) \quad -(12)$$

$$= \underline{q}^b \cdot \left(\underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \right)$$

i.e. $\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b = \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(13)$

and eq. (11) follows, QED.