Note 254(4). The Commutator and Cartan Structure Equations.

The commutator equation is:

$$[D_\mu, D_\nu] V^\rho = -T^\lambda_{\mu\nu} D_\lambda V^\rho + R^\rho_{\ \mu\nu\sigma} V^\sigma \quad -(1)$$

in which the two Cartan structure equations are given by:

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(2)$$

and

$$R^\rho_{\ \mu\nu\sigma} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(3)$$

The Christoffel connection must be antisymmetric because if $\mu = \nu$, the commutator vanishes. It is seen that the two structure equations are generated simultaneously so torsion and curvature always exist.

The Cartan structure equations are defined by the tetrad postulate:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda \quad -(4)$$

so:

$$\Gamma^a_{\mu\nu} = \Gamma^\lambda_{\mu\nu} q^a_\lambda = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu . \quad -(5)$$

The Cartan torsion is defined by:

$$T^a_{\mu\nu} = \Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu} \quad -(6)$$

$$= \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu$$

2) This is the first Cartan Maurer structure equation:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(7)$$

It is seen that the structure equation is defined by the commutator.

The second structure equation is also defined by the commutator as follows.

Consider:

$$R^a{}_{\mu\nu\sigma} = q^a_\rho R^\rho{}_{\mu\nu\sigma} \quad -(8).$$

$$= \partial_\mu \Gamma^a_{\nu\sigma} - \partial_\nu \Gamma^a_{\mu\sigma} + \Gamma^a_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^a_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$= \partial_\mu \left(\partial_\nu q^a_\sigma + \omega^a_{\nu b} q^b_\sigma\right) - \partial_\nu \left(\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma\right)$$

$$+ \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda\right)\Gamma^\lambda_{\nu\sigma} - \left(\partial_\nu q^a_\lambda + \omega^a_{\nu b} q^b_\lambda\right)\Gamma^\lambda_{\mu\sigma}$$

Note that: $$\partial_\mu \partial_\nu q^a_\sigma = \partial_\nu \partial_\mu q^a_\sigma \quad -(9)$$

So:

$$R^a{}_{\mu\nu\sigma} = \partial_\mu \left(\omega^a_{\nu b} q^b_\sigma\right) - \partial_\nu \left(\omega^a_{\mu b} q^b_\sigma\right)$$

$$+ \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda\right)\Gamma^\lambda_{\nu\sigma} - \left(\partial_\nu q^a_\lambda + \omega^a_{\nu b} q^b_\lambda\right)\Gamma^\lambda_{\mu\sigma} \quad -(10)$$

Now relabel summation indices:

$$\lambda \longrightarrow a \quad -(11)$$

and use the Leibnitz Theorem to obtain:

$$R^a_{\mu\nu\sigma} = (\partial_\mu \omega^a_{\nu b}) q^b_\sigma - (\partial_\nu \omega^a_{\mu b}) q^b_\sigma$$
$$+ \omega^a_{\nu b} \partial_\mu q^b_\sigma - \omega^a_{\mu b} \partial_\nu q^b_\sigma$$
$$+ (\partial_\mu q^a_a + \omega^a_{\mu b} q^b_a) \Gamma^a_{\nu\sigma} \quad -(12)$$
$$- (\partial_\nu q^a_a + \omega^a_{\nu b} q^b_a) \Gamma^a_{\mu\sigma}$$

Note that:

$$q^a_a = 1 \quad -(13)$$

so

$$\partial_\mu q^a_a = 0. \quad -(14)$$

Use:

$$\partial_\mu q^b_\sigma = \Gamma^b_{\mu\sigma} - \omega^b_{\mu c} q^c_\sigma$$
$$= \Gamma^b_{\mu\sigma} - \omega^b_{\mu\sigma} \quad -(15)$$

So:

$$R^a_{\mu\nu\sigma} = (\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b}) q^b_\sigma$$
$$+ \omega^a_{\nu b} (\Gamma^b_{\mu\sigma} - \omega^b_{\mu\sigma}) - \omega^a_{\mu b} (\Gamma^b_{\nu\sigma} - \omega^b_{\nu\sigma})$$
$$+ \omega^a_{\mu b} \Gamma^b_{\nu\sigma} - \omega^a_{\nu b} \Gamma^b_{\mu\sigma}$$
$$= (\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b}) q^b_\sigma \quad -(16)$$
$$+ \omega^a_{\mu b} \omega^b_{\nu\sigma} - \omega^a_{\nu b} \omega^b_{\mu\sigma}$$

Note that:

4) $$\omega^a{}_{\mu b}\,\omega^b{}_{\nu\sigma} = \omega^a{}_{\mu c}\,\omega^c{}_{\nu b}\,q^b{}_\sigma \quad -(17)$$
$$\omega^a{}_{\nu b}\,\omega^b{}_{\mu\sigma} = \omega^a{}_{\nu c}\,\omega^c{}_{\mu b}\,q^b{}_\sigma \quad -(18)$$

So:
$$R^a{}_{\mu\nu\sigma} = \left(\partial_\mu\omega^a{}_{\nu b} - \partial_\nu\omega^a{}_{\mu b} + \omega^a{}_{\mu c}\,\omega^c{}_{\nu b} - \omega^a{}_{\nu c}\,\omega^c{}_{\mu b}\right)q^b{}_\sigma \quad -(19)$$

So:
$$R^a{}_{b\mu\nu} = R^a{}_{\mu\nu b} = q^\sigma{}_b\,R^a{}_{\mu\nu\sigma}$$
$$= \partial_\mu\omega^a{}_{\nu b} - \partial_\nu\omega^a{}_{\mu b} + \omega^a{}_{\mu c}\,\omega^c{}_{\nu b} - \omega^a{}_{\nu c}\,\omega^c{}_{\mu b} \quad -(20)$$

This is the second Cartan Maurer structure equation:
$$R^a{}_b = d\wedge\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b \quad -(21)$$

The standard model assumes incorrectly that:
$$\Gamma^\lambda_{\mu\nu} =?\ \Gamma^\lambda_{\nu\mu}. \quad -(22)$$

From eq. (1):
$$[D_\mu, D_\nu]V^\rho = -\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)D_\lambda V^\rho + R^\rho{}_{\mu\nu\sigma}V^\sigma \quad -(23)$$

5) From eq. (23) it is seen that there is a one to one relation between the commutator and the Christoffel connection:

$$\boxed{[D_\mu, D_\nu]} V^\rho = \boxed{-\Gamma^\lambda_{\mu\nu}} D_\lambda V^\rho + \ldots \quad -(24)$$

so if

$$\mu = \nu \quad -(25)$$

then

$$[D_\mu, D_\nu] V^\rho = 0 \quad -(26)$$

which means that both the torsion and curvature vanish. The error (22) means that the Einstein field equation is incorrect.