Note 254(5): Analysis of the Faraday Law of Induction

For each polarization index the Faraday law of induction is:

$$\partial_0 \tilde{F}^{01} + \partial_2 \tilde{F}^{21} + \partial_3 \tilde{F}^{31} = 0 \quad -(1)$$

$$\partial_0 \tilde{F}^{02} + \partial_1 \tilde{F}^{12} + \partial_3 \tilde{F}^{32} = 0 \quad -(2)$$

$$\partial_0 \tilde{F}^{03} + \partial_1 \tilde{F}^{13} + \partial_2 \tilde{F}^{23} = 0 \quad -(3)$$

The field tensors are:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_1 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{bmatrix} \quad -(4)$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \quad -(5)$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix} \quad -(6)$$

$$\tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & E_z & -E_y \\ -cB_y & -E_z & 0 & E_x \end{bmatrix} = \begin{bmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ cB_1 & 0 & -E_3 & E_2 \\ cB_2 & E_3 & 0 & -E_1 \end{bmatrix} \quad -(7)$$

2) Hodge duals can be illustrated in Minkowski spacetime for ease of development. In the general spacetime they involve the metric. For example:

$$\tilde{F}^{01} = \frac{1}{2}\left(\epsilon^{0123}F_{23} + \epsilon^{0132}F_{32}\right) = F_{23} \quad -(8)$$

The Levi Civita symbol is defined by:

$$\epsilon^{0123} = -\epsilon^{0132} = -\epsilon^{0213} = \epsilon^{0231} = -\epsilon^{0321} = 1$$
$$= -\epsilon^{0213} = \epsilon^{2013} = -\epsilon^{2103} = \epsilon^{1203} \quad -(9)$$
$$\epsilon^{0312} = 1 = -\epsilon^{3012} = -\epsilon^{1302} = \epsilon^{0231} = -\epsilon^{2031}$$

Eq. (1) is:

$$-\frac{\partial B_X}{\partial t} - \frac{\partial E_Z}{\partial Y} + \frac{\partial E_Y}{\partial Z} = 0. \quad -(10)$$

Note that:

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ E_X & E_Y & E_Z \end{vmatrix} \quad -(11)$$

$$= \left(\frac{\partial E_Z}{\partial Y} - \frac{\partial E_Y}{\partial Z}\right)\underline{i} - \left(\frac{\partial E_Z}{\partial X} - \frac{\partial E_X}{\partial Z}\right)\underline{j} + \left(\frac{\partial E_Y}{\partial X} - \frac{\partial E_X}{\partial Y}\right)\underline{k}$$

So eq. (10) is:

$$\frac{\partial B_X}{\partial t} + \left(\underline{\nabla} \times \underline{E}\right)_X = 0 \quad -(12)$$

Considering eqs. (2) and (3) give:

$$\frac{\partial \underline{B}^a}{\partial t} + \underline{\nabla} \times \underline{E}^a = \underline{0} \quad -(13)$$

Using the Hodge dual definition it is found that eqs. (1) to (3) are:

$$\boxed{\begin{aligned} \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} &= 0 \quad -(14) \\ \partial_0 F_{31} + \partial_1 F_{03} + \partial_3 F_{10} &= 0 \quad -(15) \\ \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} &= 0 \quad -(16) \end{aligned}}$$

Note $(0, 2, 3)$, $(0, 3, 1)$ and $(0, 1, 2)$ occur in cyclic permutation. Eq. (14) to (16) are examples of

$$\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} = 0 \quad -(17)$$

i.e.

$$d \wedge F = 0 \quad -(18)$$

Eq. (17) is in tensor notation, eq. (18) is in form notation. In ECE theory they become:

$$d \wedge F^a = 0 \quad -(19)$$

$$\omega^a{}_b \wedge F^b = A^{(0)} q^b \wedge R^a{}_b \quad -(20)$$

and the Cartan identity:

$$d \wedge F^a + \omega^a{}_b \wedge F^b := A^{(0)} q^b \wedge R^a{}_b \quad -(21)$$

4) The Gauss law:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(22)$$

is given by eq. (17) with (1, 2, 3) in cyclic permutation

As shown in notes 254(1) and 254(2), Eq (17) for $\mu, \nu, \rho = 1, 2, 3$ can be expressed as:

$$\boxed{\underline{\nabla} \cdot \underline{A}^b \times \underline{\omega}^a{}_b = 0} \quad -(23)$$

which implies eq. (22), Q.E.D.

Eq. (23) is an elegant and simple form of the Cartan identity. It is equivalent to:

$$\partial_1 F^a_{23} + \partial_2 F^a_{31} + \partial_3 F^a_{12} = 0 \quad -(24)$$

which can be written as:

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b) = 0. \quad -(25)$$

Using the result:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^a = 0 \quad -(26)$$

it is seen that eqs. (23) and (25) are the same, Q.E.D.

The Faraday and Gauss laws are known as the homogeneous field equations, and are both

→ part of the same cyclic structure (17), the anti-symmetrized tensor product of a one form and two form. This suggests that the Faraday law of induction and Gauss law can be written in the same way in vector notation.

The Gauss law in the format (24) can be written as:

$$\underline{\nabla}_1 \cdot \underline{\nabla}_2 \times \underline{A}_3 + \underline{\nabla}_3 \cdot \underline{\nabla}_1 \times \underline{A}_2 + \underline{\nabla}_2 \cdot \underline{\nabla}_3 \times \underline{A}_1 = 0 \quad -(27)$$

for each polarization index a. Here:

$$\partial_1 F_{23} = \underline{\nabla}_1 \cdot \underline{\nabla}_2 \times \underline{A}_3 \quad -(28)$$

et cyclicum

Now introduce this notation for eqs. (14) to (16), i.e.

$$\partial_0 F_{23} = \underline{\nabla}_0 \cdot \underline{\nabla}_2 \times \underline{A}_3 \quad -(29)$$

$$\partial_2 F_{30} = \underline{\nabla}_2 \cdot \underline{\nabla}_3 \times \underline{A}_0 \quad -(30)$$

$$\partial_3 F_{02} = \underline{\nabla}_3 \cdot \underline{\nabla}_0 \times \underline{A}_2 \quad -(31)$$

So:

$$\underline{\nabla}_0 \cdot \underline{\nabla}_2 \times \underline{A}_3 = -\frac{\partial B_X}{\partial t} \quad -(32)$$

$$\underline{\nabla}_2 \cdot \underline{\nabla}_3 \times \underline{A}_0 = -\frac{\partial E_Z}{\partial Y} \quad -(33)$$

$$\underline{\nabla}_3 \cdot \underline{\nabla}_0 \times \underline{A}_2 = \frac{\partial E_Y}{\partial Z} \quad -(34)$$

6) The meaning of the operators is as follows:

$$\underline{\nabla}_0 := \frac{\partial}{\partial t} \quad -(35)$$

$$\underline{\nabla}_2 \times \underline{A}_3 = -B_x \, \underline{i} \quad -(36)$$

$$\underline{\nabla}_2 := \frac{\partial}{\partial Y} \underline{k} \quad -(37)$$

$$\underline{\nabla}_3 \times \underline{A}_0 = -\underline{\nabla}_0 \times \underline{A}_3 \quad -(38)$$

and

$$\underline{E} = \underline{\nabla}_0 \times \underline{A} = (\partial_0 A_1 - \partial_1 A_0)\underline{i} + (\partial_0 A_2 - \partial_2 A_0)\underline{j} + (\partial_0 A_3 - \partial_3 A_0)\underline{k} \quad -(39)$$

so:

$$\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial A}{\partial t} := \underline{\nabla}_0 \times \underline{A}} \quad -(40)$$

Therefore the Faraday law is:

$$\boxed{\underline{\nabla}_0 \cdot \underline{\nabla} \times \underline{A} + \underline{\nabla} \times (\underline{\nabla}_0 \times \underline{A}) = 0} \quad -(41)$$

and is a useful format of the Cartan identity.