254(3): Vectorial Format for Hodge Duals

Consider the Cartan identity:

$$D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} + D_\nu T^a{}_{\rho\mu} := R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \quad -(1)$$

Its Hodge dual identity is:

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}_\mu{}^{a\mu\nu} \quad -(2)$$

Therefore the space part of eq. (1) is Hodge transformed as follows:

$$-(3)$$

$$D_1 T^a{}_{23} + D_3 T^a{}_{12} + D_2 T^a{}_{31} := R^a{}_{123} + R^a{}_{312} + R^a{}_{231}$$

$$\downarrow$$

$$D_1 \tilde{T}^{a10} + D_2 \tilde{T}^{a20} + D_3 \tilde{T}^{a30} := \tilde{R}_1{}^{a10} + \tilde{R}_2{}^{a20} + \tilde{R}_3{}^{a30} \quad -(4)$$

In vector notation:

$$\boxed{\begin{array}{c} \underline{\nabla} \cdot \underset{\sim}{T}^a + \underline{\omega}^a{}_b \cdot \underset{\sim}{T}^b := \underline{V}^b \cdot \underline{R}^a{}_b \\ \downarrow \\ \underline{\nabla} \cdot \underline{\tilde{T}}^a + \underline{\omega}^a{}_b \cdot \underline{\tilde{T}}^b = \underline{V}^b \cdot \underline{\tilde{R}}^a{}_b \end{array}} \quad \begin{array}{c} -(5) \\ \\ -(6) \end{array}$$

The reverse process is:

$$D_\mu \tilde{T}^a{}_{\nu\rho} + D_\rho \tilde{T}^a{}_{\mu\nu} + D_\nu \tilde{T}^a{}_{\rho\mu} := \tilde{R}^a{}_{\mu\nu\rho} + \tilde{R}^a{}_{\rho\mu\nu} + \tilde{R}^a{}_{\nu\rho\mu} \quad -(7)$$

$$\downarrow$$

$$D_\mu T^{a\mu\nu} := R_\mu{}^{a\mu\nu} \quad -(8)$$

2) i.e.

$$\underline{\nabla} \cdot \underline{\tilde{T}}^a + \underline{\omega}^a{}_b \cdot \underline{\tilde{T}}^b = \underline{q}^b \cdot \underline{\tilde{R}}^a{}_b \quad -(9)$$

$$\downarrow$$

$$\underline{\nabla} \cdot \underline{T}^a + \underline{\omega}^a{}_b \cdot \underline{T}^b + \underline{q}^b \cdot \underline{R}^a{}_b \quad -(10)$$

For each index a define the electromagnetic field tensor as:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \quad -(11)$$

Then eqs. (5) and (6) produce:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(12)$$

with the condition:

$$\underline{q}^b \cdot \underline{R}^a{}_b = \underline{\omega}^a{}_b \cdot \underline{T}^b \quad -(13)$$

and eqs. (9) and (10) produce:

$$\underline{\nabla} \cdot \underline{E}^a = \frac{\rho}{\epsilon_0} \quad -(14)$$

As shown in notes 254(1) and 254(2), eq. (5) reduces to:

$$\boxed{\underline{\nabla} \cdot \underline{q}^b \times \underline{\omega}^a{}_b = 0} \quad -(15)$$

and eq. (13) also reduces to eq. (15). The magnetic field is defined by:

3)

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(16)$$

with:

$$\underline{A}^a = A^{(0)} \underline{q}^a \quad -(17)$$

so eq. (15) ~~means~~ that:

$$\boxed{\underline{B}^a = \underline{\nabla} \times \underline{A}^a} \quad -(18)$$

Therefore:

$$\underline{\nabla} \cdot \underline{B}^a = \underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^a = 0 \quad -(19)$$

which is eq. (12), Q.E.D.

It is important to note that both eqs. (5) and (6) describe the magnetic field, and both eqs. (9) and (10) describe the electric field. This result is by definition (11). The space part of (11) is considered in eq. (5) and this is magnetic. The Hodge dual in eq. (6) also refers to the magnetic field. In eq. (9) the space part of the Hodge dual of eq. (11) is being considered. The Hodge dual of eq. (11) with raised indices is, for each a:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix} \quad -(20)$$

4) So eqns. (9) and (10) give again eq. (15). However the electric field is defined by:

$$\underline{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} - \omega^a{}_{0b}\underline{A}^b + \underline{\omega}^a{}_b\phi^b \quad -(21)$$

So eq. (15) does not imply that the divergence of $\underline{E}^a$ is zero in general. So we arrive at eq. (14) in general.

The next note will continue this development for $\mu = 1, 2, 3$ in eqs. (7) or (8).