254(1): Translation from Differential Form to Vector Notation.

Consider the wedge product of two one-forms:

$$(A \wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu \quad -(1)$$

in four dimensions 0, 1, 2, 3. The space part of eq. (1) is represented by:

$$i, j, k = 1, 2, 3. \quad -(2)$$

and is equivalent to the vector product:

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\underline{i} - (A_x B_z - A_z B_x)\underline{j} + (A_x B_z - A_z B_y)\underline{k} \quad -(3)$$

Similarly

$$d \wedge A \rightarrow \underline{\nabla} \times \underline{A} \quad -(4)$$

for the three space dimensions defined by eq. (2). In tensor notation

$$d \wedge A = \partial_\mu A_\nu - \partial_\nu A_\mu \quad -(5)$$

in four dimensions. If one of the indices in eq. (5) is 0 then there are terms such as:

$$(d \wedge A)_{0i} = \partial_0 A_1 - \partial_1 A_0 \quad -(6)$$

In vector notation it is possible to denote eq. (6) by:

$$\boxed{\underline{\nabla}_0 \times \underline{A} = (\partial_0 A_1 - \partial_1 A_0)\underline{i} + (\partial_0 A_2 - \partial_2 A_0)\underline{j} + (\partial_0 A_3 - \partial_3 A_0)\underline{k}} \quad -(7)$$

2) Similarly it is possible to adapt the notation:

$$\underline{A_0} \times \underline{B} = (A_0 B_1 - B_1 A_0)\,\underline{i} + (A_0 B_2 - B_2 A_0)\,\underline{j} + (A_0 B_3 - B_3 A_0)\,\underline{k} \quad -(8)$$

In this vector notation the first and second Cartan structure equations and the Cartan identity become easier to understand.

First Cartan Structure Equation.

In differential form notation this is:

$$T = D \wedge q = d \wedge q + \omega \wedge q \quad -(9)$$

which is shorthand for:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b. \quad -(10)$$

In tensor notation:

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu \quad -(11)$$

Therefore the space part of eq. (11) is:

$$\underline{T}^a{}_{spin} = \underline{\nabla} \times \underline{q}^a + \underline{\omega}^a{}_b \times \underline{q}^b \quad -(12)$$

as given in the ECE engineering model. Eq. (12) defines the spin-torsion vector. The time part

3) of eq. (11) can be written as:

$$\underline{T}^a_{orb} = \underline{\nabla}_0 \times \underline{q}^a + \underline{\omega}^a_{0b} \times \underline{q}^b \quad -(13)$$

and defines the orbital torsion vector. Eqs. (12) and (13) translate into definitions of the electric and magnetic fields or gravitational and magnetogravitational fields.

Second Cartan Structure Equation

In differential form notation this is:

$$R = D \wedge \omega = d \wedge \omega + \omega \wedge \omega \quad -(14)$$

which is shorthand for:

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \quad -(15)$$

In tensor notation:

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{\mu c}\, \omega^c{}_{\nu b} - \omega^a{}_{\nu c}\, \omega^c{}_{\mu b}. \quad -(16)$$

In vector notation:

$$\underline{R}^a{}_{b\,spin} = \underline{\nabla} \times \underline{\omega}^a{}_b + \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(17)$$

$$\underline{R}^a{}_{b\,orbital} = \underline{\nabla}_0 \times \underline{\omega}^a{}_b + \underline{\omega}^a{}_{0c} \times \underline{\omega}^c{}_b \quad -(18)$$

The vector notation is much easier to grasp than the form notation. The tensor notation can become very cumbersome.

4)

The Cartan Identity

In form notation this is:

$$D \wedge T := R \wedge q \quad - (19)$$

One side is identical with the other. This is not easy to prove but the proof with full detail is given in the UFT papers. Eq. (19) is short hand for:

$$d \wedge T^a + \omega^a{}_b \wedge T^b := R^a{}_b \wedge q^b \quad - (20)$$

In tensor notation eq. (20) must be written out using the rule for the wedge product of a one form with a two form:

$$A_\mu \wedge B_{\nu\rho} = A_\mu B_{\nu\rho} + A_\rho B_{\mu\nu} + A_\nu B_{\rho\mu} \quad - (21)$$

This is an antisymmetrized tensor product.

Therefore eq. (20) is:

$$\begin{aligned} &\partial_\mu T^a{}_{\nu\rho} + \partial_\rho T^a{}_{\mu\nu} + \partial_\nu T^a{}_{\rho\mu} \\ &+ \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\rho b} T^b{}_{\mu\nu} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} \qquad - (22) \\ &= q^b_\mu R^a{}_{b\nu\rho} + q^b_\rho R^a{}_{b\mu\nu} + q^b_\nu R^a{}_{b\rho\mu} \end{aligned}$$

where T and R are defined by the structure equations.

Eq. (22) can be written as an equation in Hodge duals as in the UFT papers:

5)

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}_\mu{}^{a\mu\nu} \quad -(23)$$

This equation is useful to give the field equations as in "Criticisms of the Einstein Field Equation" and many other papers and books on www.aias.us.

In this note we wish to explore the structure of the Cartan identity in a different way. In the most condensed notation it is:

$$d \wedge T + \omega \wedge T := R \wedge q \quad -(24)$$

where

$$T = d \wedge q + \omega \wedge q \quad -(25)$$

$$R = d \wedge \omega + \omega \wedge \omega \quad -(26)$$

So:

$$d \wedge (d \wedge q + \omega \wedge q) + \omega \wedge (d \wedge q + \omega \wedge q)$$

$$= q \wedge (d \wedge \omega + \omega \wedge \omega) \quad -(27)$$

Although elegant this is too abstract to be immediately useful. A more useful form of it is given in vector notation. First consider the wedge product:

$$A_\mu \wedge D_{\nu\rho} = A_\mu D_{\nu\rho} + A_\rho D_{\mu\nu} + A_\nu D_{\rho\mu} \quad -(28)$$

where:

$$D_{\nu\rho} = B_\nu C_\rho - B_\rho C_\nu \quad -(29)$$

$$D_{\mu\nu} = B_\mu C_\nu - B_\nu C_\mu \quad -(30)$$

$$D_{\rho\mu} = B_\rho C_\mu - B_\mu C_\rho \quad -(31)$$

6)

For example:

$$A_\mu \wedge D_{\nu\rho} = A_\mu (B_\nu C_\rho - B_\rho C_\nu) + \cdots \quad -(32)$$

For space like indices there are terms such as:

$$A_1 \wedge D_{23} = A_1 (B_2 C_3 - B_3 C_2) + \cdots \quad -(33)$$

$$= A_1 D_1 + \cdots$$

So in vector notation the product (21) translates to:

$$A_\mu \wedge D_{\nu\rho} \rightarrow \underline{A} \cdot \underline{B} \times \underline{C} \quad -(34)$$

Therefore the spacelike part of eq. (22) becomes:

$$\boxed{\begin{aligned} (\underline{\nabla} + \underline{\omega}^a{}_b) \cdot (\underline{\nabla} \times \underline{q}^b + \underline{\omega}^b{}_c \times \underline{q}^c) &:= \underline{q}^b \cdot \underline{R}^a{}_b \\ &:= \underline{q}^b \cdot (\underline{\nabla} \times \underline{\omega}^a{}_b + \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b) \end{aligned}} \quad -(35)$$

This is the spin Cartan identity, or spacelike Cartan identity, in vector notation.

Now note that:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{q}^b = 0 \quad -(36)$$

and:

$$\underline{\nabla} \cdot \underline{a} \times \underline{b} = \underline{b} \cdot \underline{\nabla} \times \underline{a} - \underline{a} \cdot \underline{\nabla} \times \underline{b} \quad -(37)$$

So eq. (35) simplifies to:

7)

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b + \underline{\omega}^a{}_b \cdot \underline{\omega}^b{}_c \times \underline{q}^c := \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b + \underline{q}^b \cdot \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad - (38)$$

Now use:

$$\underline{\omega}^a{}_b \cdot \underline{\omega}^b{}_c \times \underline{q}^c = \underline{q}^c \cdot \left(\underline{\omega}^a{}_b \times \underline{\omega}^b{}_c \right) \quad - (39)$$

$$= \underline{q}^b \cdot \left(\underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \right)$$

after relabelling of dummy indices. So eq. (38) becomes:

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b := \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad - (40)$$

Finally use eq. (37) to find:

$$\boxed{\underline{\nabla} \cdot \underline{q}^b \times \underline{\omega}^a{}_b := 0} \quad - (39)$$

which is a new form of the Cartan identity.

As in the engineering model the magnetic field is defined as: — (40)

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a + \underline{A}^b \times \underline{\omega}^a{}_b$$

so eq. (39) means:

$$\boxed{\underline{\nabla} \cdot \underline{B}^a = 0} \quad - (41)$$