254(6): Simplified Format of the Cartan and Evans Identities

Consider the four dimensional curl in tensor notation:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad -(1)$$

In the U(1) or Maxwell Heaviside theory this means:

$$
\begin{aligned}
cB_3 &= F_{12} = \partial_1 A_2 - \partial_2 A_1 \\
-cB_2 &= F_{13} = \partial_1 A_3 - \partial_3 A_1 \\
cB_1 &= F_{23} = \partial_2 A_3 - \partial_3 A_2 \\
-E_1 &= F_{01} = \partial_0 A_1 - \partial_1 A_0 \\
-E_2 &= F_{02} = \partial_0 A_2 - \partial_2 A_0 \\
-E_3 &= F_{03} = \partial_0 A_3 - \partial_3 A_0
\end{aligned}
\quad -(2)
$$

In vector notation:

$$\underline{\nabla} \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{0} & \underline{0} \\ 0 & \partial_2 & \partial_3 \\ 0 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \underline{0} & \underline{j} & \underline{0} \\ \partial_1 & 0 & \partial_3 \\ A_1 & 0 & A_3 \end{vmatrix} + \begin{vmatrix} \underline{0} & \underline{0} & \underline{k} \\ \partial_1 & \partial_2 & 0 \\ A_1 & A_2 & 0 \end{vmatrix}$$

$$= \underline{i}\left(\partial_2 A_3 - \partial_3 A_2\right) - \underline{j}\left(\partial_1 A_3 - \partial_3 A_1\right) + \underline{k}\left(\partial_1 A_2 - \partial_2 A_1\right) \quad -(3)$$

$$\left(\underline{\nabla} \times \underline{A}\right)_{HD} = \begin{vmatrix} \underline{i} & \underline{0} & \underline{0} \\ 0 & \partial_0 & \partial_1 \\ 0 & A_0 & A_1 \end{vmatrix} + \begin{vmatrix} \underline{0} & \underline{j} & \underline{0} \\ \partial_2 & 0 & \partial_0 \\ A_2 & 0 & A_0 \end{vmatrix} + \begin{vmatrix} \underline{0} & \underline{0} & \underline{k} \\ \partial_0 & \partial_3 & 0 \\ A_0 & A_3 & 0 \end{vmatrix}$$

$$= \underline{i}\left(\partial_0 A_1 - \partial_1 A_0\right) + \left(\partial_0 A_2 - \partial_2 A_0\right)\underline{j} + \left(\partial_0 A_3 - \partial_3 A_0\right)\underline{k} \quad -(4)$$

where

$$\underline{\nabla}_0 \times \underline{A} := \left(\underline{\nabla} \times \underline{A} \right)_{HD} \quad -(5)$$

is the Hodge dual of $\underline{\nabla} \times \underline{A}$. The components are:

$$\partial^0 A^1 - \partial^1 A^0 = \epsilon^{0123} \left(\partial_2 A_3 - \partial_3 A_2 \right) \quad -(6)$$

$$\partial^0 A^2 - \partial^2 A^0 = \epsilon^{0213} \left(\partial_1 A_3 - \partial_3 A_1 \right) \quad -(7)$$

$$\partial^0 A^3 - \partial^3 A^0 = \epsilon^{0312} \left(\partial_1 A_2 - \partial_2 A_1 \right) \quad -(8)$$

where

$$\epsilon^{0123} = \epsilon^{0213} = \epsilon^{0312} = 1 \quad -(9)$$

Therefore the four dimensional curl is the usual curl $\underline{\nabla} \times \underline{A}$ and its Hodge dual.

This result is very useful for the simplification of the Cartan and Evans identities. First note that if

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A} = 0 \quad -(10)$$

then

$$\underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{A} \right)_{HD} \neq 0 \quad -(11)$$

Eq. (10) is a type of Jacobi identity:

$$\partial_3 \left(\partial_1 A_2 - \partial_2 A_1 \right) + \partial_1 \left(\partial_2 A_3 - \partial_3 A_2 \right) + \partial_2 \left(\partial_3 A_1 - \partial_1 A_3 \right) = 0 \quad -(12)$$

However:

3)

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A})_{HD} = \partial_1 (\partial_0 A_1 - \partial_1 A_0) + \partial_2 (\partial_0 A_2 - \partial_2 A_0) + \partial_3 (\partial_0 A_3 - \partial_3 A_0)$$

$$= - \underline{\nabla} \cdot \left(\underline{\nabla} \phi + \frac{\partial \underline{A}}{\partial t} \right) \quad -(13)$$

$$= - \nabla^2 \phi - \underline{\nabla} \cdot \frac{\partial \underline{A}}{\partial t}$$

$$\neq 0$$

QED. Therefore we arrive at the elegant result:

$$\boxed{\begin{aligned} \underline{E} &= (\underline{\nabla} \times \underline{A})_{HD} = - \underline{\nabla} \phi - \frac{\partial \underline{A}}{\partial t} \\ \text{and} \quad \underline{B} &= \underline{\nabla} \times \underline{A} \end{aligned}} \quad -(14)$$

The electric field strength $\underline{E}$ in volts m^{-1} is the Hodge dual of the magnetic flux density $\underline{B}$ in tesla.

In note 254 (1) it was shown that the Cartan identity is:

$$\boxed{\underline{\nabla} \cdot \underline{q}^b \times \underline{\omega}^a{}_b := 0} \quad -(15)$$

4)

Eq. (15) is a very elegant result which greatly reduces the complexity of the tensorial format of the Cartan identity. Eq. (15) is also a Jacobi identity. This follows from a comparison of eqs. (10) and (15). In ECE theory eq. (15) is:

$$\underline{\nabla} \cdot \underline{A}^b \times \underline{\omega}^a{}_b := 0 \quad -(16)$$

The magnetic flux density in ECE theory is:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a + \underline{A}^b \times \underline{\omega}^a{}_b \quad -(17)$$

so eq. (16) implies:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(18)$$

It follows that:

$$\underline{\nabla} \cdot \left(\underline{\omega}^a{}_b \times \underline{A}^b\right)_{HD} \neq 0 \quad -(19)$$

where

$$\boxed{\left(\underline{\omega}^a{}_b \times \underline{A}^b\right)_{HD} = -\omega^a{}_{0b}\underline{A}^b + \underline{\omega}^a{}_b \phi^b} \quad -(20)$$

The electric field strength in ECE theory is:

$$\underline{E}^a = \quad -(21)$$

$$= \left(\underline{\nabla} \times \underline{A}^a\right)_{HD} - \left(\underline{\omega}^a{}_b \times \underline{A}^b\right)_{HD}$$

5) and

$$\underline{\nabla} \cdot \underline{E}^a = \frac{\rho^a}{\epsilon_0} \quad -(22)$$

Results in ECE Theory

Electric Field Strength

$$\underline{E}^a = \left(\underline{\nabla} \times \underline{A}^a\right)_{HD} + \left(\underline{A}^a \times \underline{\omega}^a{}_b\right)_{HD}$$

$$= -\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} - \omega^a{}_{ob}\underline{A}^b + \underline{\omega}^a{}_b\phi^b$$

Magnetic Flux Density

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a + \underline{A}^b \times \underline{\omega}^a{}_b$$

Cartan Identity

$$\underline{\nabla} \cdot \underline{q}^b \times \underline{\omega}^a{}_b := 0$$

and

$$\underline{\nabla} \cdot \left(\underline{q}^b \times \underline{\omega}^a{}_b\right)_{HD} \neq 0$$

Similarly the gravitomagnetic field is the Hodge dual of the gravitational field.