253(4): Higher Order Terms in the Fermi Equation.

To date in this series of recent papers the minimal prescription has been used as follows:

$$E \rightarrow E - e\phi \quad -(1)$$

where ϕ is the electromagnetic scalar potential. The effect of the gravitational field can be added as follows:

$$E \rightarrow E - e\phi + m\Phi, \quad -(2)$$

where Φ is the gravitational potential. Here, the charge on the electron is $-e$ and the mass of the electron is m. These equations are sufficient to describe the H atom in a gravitational field. The starting equation as usual is the Einstein energy equation:

$$E^2 = c^2p^2 + m^2c^4, \quad -(3)$$

which is a reexpression of the relativistic momentum:

$$\underline{p} = \gamma m \underline{v}. \quad -(4)$$

In the usual minimal prescription (1), eq. (3) becomes:

$$(E - e\phi)^2 = c^2p^2 + m^2c^4 \quad -(5)$$

which can be rewritten as:

$$(E - e\phi)^2 - m^2c^4 = c^2p^2 \quad -(6)$$

$$= (E - e\phi - mc^2)(E - e\phi + mc^2)$$

This is a very useful factorization of eq. (5).

2) It allows the total energy to be written as:

$$E = e\phi + mc^2 + \frac{c^2 p^2}{E + mc^2 - e\phi} \quad -(7)$$

as in recent notes. In the usual procedure the E in the denominator on the right hand side is approximated by:

$$E = \gamma mc^2 \sim mc^2, \quad -(8)$$

i.e by

$$v <<< c. \quad -(9)$$

The approximation (8) is the one responsible for the deduction of the g factor of the electron and for the Thomas factor.

From eqs. (7) and (8):

$$E = e\phi + mc^2 + \frac{c^2 p^2}{2mc^2 - e\phi}. \quad -(10)$$

At this point the Pauli matrices are introduced as follows:

$$E = e\phi + mc^2 + c^2 \underline{\sigma} \cdot \underline{p} \frac{1}{2mc^2 - e\phi} \underline{\sigma} \cdot \underline{p} \quad -(11)$$

The placement of $(2mc^2 - e\phi)^{-1}$ is made in this way in preparation for quantization:

$$\underline{p} = -i\hbar \underline{\nabla} \quad -(12)$$

3) There are several subtleties like this which are almost never explained well in sites and textbooks. These must have been introduced by Dirac originally. Eq. (11) is rewritten as:

$$E = e\phi + mc^2 + \frac{1}{2m} \underline{\sigma} \cdot \underline{p} \left(1 - \frac{e\phi}{2mc^2} \right)^{-1} \underline{\sigma} \cdot \underline{p} \quad -(13)$$

Up to now, everything is classical. The total energy on the left hand side is the hamiltonian H.

In the usual procedure it is assumed that:

$$x = \frac{e\phi}{2mc^2} << 1 \quad -(14)$$

In this approximation:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \quad -(15)$$

Dirac truncated this expansion at first order:

$$(1-x)^{-1} \sim 1 + x \quad -(16)$$

to produce the g factor of the electron, the Landé factor, the Thomas factor, spin orbit coupling spectra, ESR and NMR. In recent papers we have used eq. (16) to produce several novel results.

If, however, we use:

$$(1-x)^{-1} = 1 + x + x^2 \quad -(17)$$

4) eq. (13) becomes:

$$E = e\phi + mc^2 + \frac{1}{2m}\underline{\sigma}\cdot\underline{p}\left(1 + \frac{e\phi}{2mc^2} + \frac{e^2\phi^2}{4m^2c^4}\right)\underline{\sigma}\cdot\underline{p}$$

– (18)

There is a new type of spin-orbit spectroscopy described by:

$$E_1 = \frac{e^2}{8m^3c^4}\underline{\sigma}\cdot\underline{p}\,\phi^2\,\underline{\sigma}\cdot\underline{p} \quad -(19)$$

This quantizes to:

$$H_1\psi = -\frac{ie^2\hbar}{8m^3c^4}\underline{\sigma}\cdot\underline{\nabla}\left(\phi^2\,\underline{\sigma}\cdot\underline{p}\,\psi\right) \quad -(20)$$

$$= -\frac{ie^2\hbar}{8m^3c^4}\left(\underline{\sigma}\cdot\underline{\nabla}\phi^2\,\underline{\sigma}\cdot\underline{p}\right)\psi$$

$$+ \cdots$$

in which:

$$\phi^2 = \frac{e^2}{16\pi^2\epsilon_0^2 r^2}, \quad -(21)$$

– (22)

$$\underline{\nabla}\phi^2 = -\frac{e^2}{8\pi^2\epsilon_0^2 r^3} = -\frac{e^2}{8\pi^2\epsilon_0^2}\frac{\underline{r}}{r^4}$$

So:

$$H_1\psi = \frac{ie^4\hbar}{64\pi^2\epsilon_0^2 m^3c^4r^4}\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{p}\,\psi$$

– (23)

) Using the Pauli algebra:

$$\underline{\sigma} \cdot \underline{r} \, \underline{\sigma} \cdot \underline{p} = \underline{r} \cdot \underline{p} + i \, \underline{\sigma} \cdot \underline{L} \quad -(24)$$

gives:

$$\text{Real } H_1 \phi = - \frac{e^4 \hbar}{64 \pi^2 \epsilon_0^2 m^3 c^4 r^4} \, \underline{\sigma} \cdot \underline{L} \, \phi \quad -(25)$$

$$= - \frac{e^4 \hbar^2}{64 \pi^2 \epsilon_0^2 m^3 c^4 r^4} \left(j(j+1) - l(l+1) - s(s+1) \right) \phi$$

The energy eigenvalues are:

$$E_1 = - \frac{e^4 \hbar^2}{64 \pi^2 \epsilon_0^2 m^3 c^4} \left(j(j+1) - l(l+1) - s(s+1) \right) \int \frac{\phi^* \phi}{r^4} d\tau$$

$$= - \left(\frac{e^2 \hbar}{8 \pi \epsilon_0 m c^2} \right)^2 \frac{\left(j(j+1) - l(l+1) - s(s+1) \right)}{m c^2} \int \frac{\phi^* \phi}{r^4} d\tau \quad -(26)$$

This compares with the first order result:

$$E_0 = \frac{e^2 \hbar^2}{16 \pi \epsilon_0 m^2 c^2} \left(j(j+1) - l(l+1) - s(s+1) \right) \int \frac{\phi^* \phi}{r^3} d\tau \quad -(27)$$

The result (26) can be written as:

$$E_1 = -m\left(\frac{e^2 \hbar}{8\pi \epsilon_0 m^2 c^2}\right)^2 \left(j(j+1) - l(l+1) - s(s+1)\right) \int \frac{\psi^* \psi}{r^4} d\tau$$

The units of $e^2 \hbar^2 / (16\pi \epsilon_0 m^2 c^2 r^3)$ are joules, and the units of $e^4 \hbar^2 / (64\pi^2 \epsilon_0^2 m^3 c^4 r^4)$ are also joules. The ratio of the second order to first order effect is: — (28)

$$\left|\frac{E_1}{E_0}\right| = \frac{e^2}{4\pi \epsilon_0 m c^2} \int \frac{\psi^* \psi}{r^4} d\tau \Big/ \int \frac{\psi^* \psi}{r^3} d\tau \quad -(29)$$

where

$$\left\langle \frac{1}{r^4} \right\rangle = \int \frac{\psi^* \psi}{r^4} d\tau \quad -(30)$$

$$\left\langle \frac{1}{r^3} \right\rangle = \int \frac{\psi^* \psi}{r^3} d\tau \quad -(31)$$

Therefore:

$$\boxed{\left|\frac{E_1}{E_0}\right| = 2.824 \times 10^{-15} \left\langle \frac{1}{r^4} \right\rangle \Big/ \left\langle \frac{1}{r^3} \right\rangle} \quad -(32)$$

Harris illustrated the method of calculation, the effect of gravitation is considered by:

$$e\phi \rightarrow e\phi - m\Phi \quad -(33)$$

in eq. (18). Eq. (33) in eq. (18) produces:

$$E = e\phi - m\Phi + mc^2 + \frac{1}{2m}\underline{\sigma}\cdot\underline{p}\left(1 + \frac{e\phi - m\Phi}{2mc^2} + \frac{(e\phi - m\Phi)^2}{4m^2c^4}\right)\underline{\sigma}\cdot\underline{p} \quad -(33)$$

The first order effect was considered in the previous note. The second order term is:

$$E_2 = \frac{1}{8m^3c^4}\underline{\sigma}\cdot\underline{p}\,(e\phi - m\Phi)^2\,\underline{\sigma}\cdot\underline{p} \quad -(34)$$

In the rest of this note we evaluate the mixed term:

$$E_3 = -\frac{em}{4m^3c^4}\underline{\sigma}\cdot\underline{p}\,\phi\Phi\,\underline{\sigma}\cdot\underline{p} \quad -(35)$$

$$= -\frac{e}{4m^2c^4}\underline{\sigma}\cdot\underline{p}\,\phi\Phi\,\underline{\sigma}\cdot\underline{p}$$

In order to consider the effect of the earth's gravitation on the spin-orbit spectrum of an H atom we use:

$$\phi = -\frac{e}{4\pi\epsilon_0 r}, \quad \Phi = -\frac{GM}{R} \quad -(36)$$

where $\underline{r}$ is the distance between the proton and the centre of the H atom and $\underline{R}$ is the distance between the electron and the centre of the earth.

5) Quantization of eq. (35) produces:

$$\hat{H}_3 \phi = \frac{ie\hbar}{4m^2c^4} \underline{\sigma} \cdot \underline{\nabla} \cdot (\phi \underline{\Phi}) \underline{\sigma} \cdot \underline{p} \phi \quad -(37)$$

It can be checked that the right hand side has the correct units of joules. Here:

$$\phi \underline{\Phi} = \frac{e}{4\pi\epsilon_0 r} \frac{G\underline{M}}{R} \quad -(38)$$

So

$$\hat{H}_3 \phi = \frac{ie^2\cancel{\hbar}}{16\pi\epsilon_0 m^2 c^4} \underline{\sigma} \cdot \underline{\nabla} \left(\frac{\underline{\Phi}}{r} \right) \underline{\sigma} \cdot \underline{p} \phi$$

$$= \frac{ie^2\hbar \underline{M} G}{16\pi\epsilon_0 m^2 c^4} \underline{\sigma} \cdot \underline{\nabla} \left(\frac{1}{rR} \right) \underline{\sigma} \cdot \underline{p} \phi + \dots \quad -(39)$$

Using the Leibnitz Theorem:

$$\underline{\nabla} \left(\frac{1}{rR} \right) = \frac{1}{R} \underline{\nabla} \left(\frac{1}{r} \right) + \frac{1}{r} \underline{\nabla} \left(\frac{1}{R} \right)$$

$$= -\frac{1}{R} \frac{\underline{r}}{r^3} - \frac{1}{r} \frac{\underline{R}}{R^3} \quad -(40)$$

Therefore:

$$\phi = \frac{-ie^2\hbar \underline{M} G}{16\pi\epsilon_0 m^2 c^4} \left(\frac{1}{Rr^3} \underline{\sigma} \cdot \underline{r} \, \underline{\sigma} \cdot \underline{p} + \frac{1}{rR^3} \underline{\sigma} \cdot \underline{R} \, \underline{\sigma} \cdot \underline{p} \right) \cancel{\phi} \quad -(41)$$

9) where:

$$\underline{\sigma} \cdot \underline{r}\, \underline{\sigma} \cdot \underline{p} = \underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L} \quad -(42)$$

$$\underline{\sigma} \cdot \underline{R}\, \underline{\sigma} \cdot \underline{p} = \underline{R} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L_1} \quad -(43)$$

where:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(44)$$

$$\underline{L_1} = \underline{R} \times \underline{p} \quad -(45)$$

The term of relevance is:

$$\text{Real } \hat{H}_3 \phi = \frac{e^2 \hbar M G}{16\pi \epsilon_0 m^2 c^4 R r^3} \underline{\sigma} \cdot \underline{L} \phi$$

$$= \frac{e^2 \hbar^2 M G}{16\pi \epsilon_0 m^2 c^4 R r^3} \left(j(j+1) - l(l+1) - s(s+1) \right) \phi \quad -(46)$$

This gives the energy expectation values:

$$E_3 = \frac{e^2 \hbar^2}{16\pi \epsilon_0 m^2 c^2} \left(\frac{MG}{Rc^2} \right) \left(j(j+1) - l(l+1) - s(s+1) \right) \int \frac{\psi^* \psi}{r^3} d\tau \quad -(47)$$

Comparison of eqns. (27) and (47) shows that:

$$\boxed{\frac{E_3}{E_0} = \frac{MG}{Rc^2}} \quad -(48)$$

10) It is seen that the effect of gravitation on the spin orbit coupling of H is $MG/(Rc^2)$. This happens to be the term that appears in the so called "Schwarzschild radius"

$$r_0 = \frac{MG}{c^2} \quad -(49)$$

For the earth:

$$r_0 = 8.87 \times 10^{-3} \, m \quad -(50)$$

and for the sun:

$$r_0 = 2.95 \times 10^{3} \, m \quad -(51)$$

The radius of the earth is:

$$R(\text{earth}) = 6.371 \times 10^{6} \, m \quad -(52)$$

and the radius of the sun is:

$$R(\text{sun}) = 6.955 \times 10^{8} \, m \quad -(53)$$

So

$$\left(\frac{E_3}{E_0}\right)_{\text{Earth}} = \frac{8.87 \times 10^{-3}}{6.371 \times 10^{6}} = 1.39 \times 10^{-9} \quad -(54)$$

$$\left(\frac{E_3}{E_0}\right)_{\text{sun}} = \frac{2.95 \times 10^{3}}{6.955 \times 10^{8}} = 4.242 \times 10^{-5} \quad -(55)$$

This means that relatively large effects are expected on the spin orbit splitting of the H atom in the atmosphere of the sun. The spectrum should be different from that in the laboratory.

In a neutron star, M is 1.4 to 3.2

11) solar masses and R is only 12 km typically,

So:

$$\left(\frac{E_3}{E_0}\right)_{\text{neutron star}} = \frac{MG}{Rc^2}$$

$$= (1.4 \text{ to } 3.2) \times \frac{2.95 \times 10^3}{1.2 \times 10^3}$$

$$= 2.458 \times (1.4 \text{ to } 3.2) \quad -(56)$$

$$= 3.442 \text{ to } 7.867$$

So in an H atom in the atmosphere of a neutron star the spin-orbit splitting should be 3.442 to 7.867 larger than in an earth laboratory.

Finally here is another type of spectrum given by the second term on the right hand side of the hamiltonian from eq. (41)

$$\text{Real } \hat{H}_{31}\psi = \frac{e^2 \hbar M G}{16\pi\epsilon_0 m^2 c^4 r R^3} \underline{\sigma} \cdot \underline{R} \times \underline{p}\psi \quad -(57)$$

and this will be developed in the next note.