3(2): Development of the Fermion Equation with Spin Connection and Gravitation

Start by considering the Einstein energy equation or equation of relativistic momentum:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(1)$$

i.e.

$$(E - mc^2)(E + mc^2) = c^2 p^2 . \quad -(2)$$

Use the minimal prescription as follows:

$$E \rightarrow E - e\phi \quad -(3)$$

where $-e$ is the charge on the electron and ϕ is the scalar potential of the electromagnetic field. Therefore:

$$E = e\phi + mc^2 + \frac{c^2 p^2}{E - e\phi + mc^2} \quad -(4)$$

In the SU(2) basis:

$$E = e\phi + mc^2 + c^2 \underline{\sigma} \cdot \underline{p} \left(\frac{1}{E - e\phi + mc^2} \right) \underline{\sigma} \cdot \underline{p} . \quad -(5)$$

In the approximation:

$$E \sim mc^2 \quad -(6)$$

eq. (5) is:

$$E = e\phi + mc^2 + \frac{1}{2m} \underline{\sigma} \cdot \underline{p} \left(1 - \frac{e\phi}{2mc^2} \right)^{-1} \underline{\sigma} \cdot \underline{p} \quad -(6)$$

Now assume:

$$e\phi \ll mc^2 \quad -(7)$$

So:

$$E = e\phi + mc^2 + \frac{1}{2m}\underline{\sigma}\cdot\underline{p}\left(1+\frac{e\phi}{2mc^2}\right)\underline{\sigma}\cdot\underline{p} \quad -(8)$$

$$= e\phi + mc^2 + \frac{1}{2m}\underline{\sigma}\cdot\underline{p}\,\underline{\sigma}\cdot\underline{p} + \frac{e}{4m^2c^2}\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p}$$

This equation quantizes to:

$$H\psi = \left(e\phi + mc^2 + \frac{1}{2m}\underline{\sigma}\cdot\underline{p}\,\underline{\sigma}\cdot\underline{p} + \frac{e}{4m^2c^2}\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p}\right)\psi \quad -(9)$$

where

$$\underline{p}\psi = -i\hbar\underline{\nabla}\psi \quad -(10)$$

Consider

$$\phi = -\frac{e}{4\pi\epsilon_0 r} \quad -(11)$$

to be the potential between electron and proton in the H atom. Then eq. (9) is the fermion equation of the H atom.

Spin orbit coupling is given by the term:

$$H_1\psi = \frac{e}{4m^2c^2}\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p}\,\psi \quad -(12)$$

$$= -\frac{ie\hbar}{4m^2c^2}\underline{\sigma}\cdot\underline{\nabla}\phi\,\underline{\sigma}\cdot\underline{p}\,\psi$$

) The hamiltonian must be interpreted as:

$$\hat{H}_1 \psi = -\frac{ie\hbar}{4m^2c^2} \underline{\sigma} \cdot \underline{\nabla} \left(\phi \, \underline{\sigma} \cdot \underline{p} \, \psi \right) \quad -(13)$$

By the Leibnitz Theorem:

$$\underline{\nabla} \left(\phi \, \underline{\sigma} \cdot \underline{p} \, \psi \right) = \underline{\nabla} \left(\underline{\sigma} \cdot \underline{p} \right) \phi \, \psi + \underline{\sigma} \cdot \underline{p} \, \underline{\nabla} \left(\phi \psi \right) \quad -(14)$$

If

$$\underline{\nabla} \left(\underline{\sigma} \cdot \underline{p} \right) = \underline{0} \quad -(15)$$

i.e. if $\underline{p}$ has no dependence on $\underline{r}$, then:

$$H_1 \psi = -\frac{ie\hbar}{4m^2c^2} \underline{\sigma} \cdot \underline{\nabla} \left(\phi \psi \right) \underline{\sigma} \cdot \underline{p} \quad -(16)$$

Using the Leibnitz Theorem:

$$\underline{\nabla} \left(\phi \psi \right) = \left(\underline{\nabla} \phi \right) \psi + \phi \left(\underline{\nabla} \psi \right) \quad -(17)$$

Therefore:

$$H_1 \psi = -\frac{ie\hbar}{4m^2c^2} \left(\underline{\sigma} \cdot \underline{\nabla} \phi \, \underline{\sigma} \cdot \underline{p} \, \psi + \phi \, \underline{\sigma} \cdot \underline{\nabla} \psi \, \underline{\sigma} \cdot \underline{p} \right) \quad -(18)$$

At this point the standard physics uses:

$$\underline{E} = -\underline{\nabla}\phi \quad -(19)$$

but ECE physics uses the spin connection and the antisymmetry law to produce an equation formally identical to eq. (19) but written as part of a unified field theory. There is the added possibility of spin connection resonance.

From eqs. (18) and (19), using eq. (11):

$$H_1\phi = \frac{ie\hbar}{4m^2c^2}\,\underline{\sigma}\cdot\underline{E}\,\underline{\sigma}\cdot\underline{p}\,\phi + \frac{ie^2\hbar}{16\pi\epsilon_0 m^2c^2 r}\,\underline{\sigma}\cdot\underline{\nabla}\phi\,\underline{\sigma}\cdot\underline{p} \quad -(20).$$

where

$$\underline{E} = -\frac{e}{4\pi\epsilon_0}\frac{\underline{r}}{r^3} \quad -(21)$$

so:

$$H_1\phi = -\frac{ie^2\hbar}{16\pi\epsilon_0 m^2c^2}\left(\frac{1}{r^3}\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{p}\,\phi - \frac{1}{r}\underline{\sigma}\cdot\underline{\nabla}\phi\,\underline{\sigma}\cdot\underline{p}\right) \quad -(22)$$

By Pauli algebra:

$$\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{p} = \underline{r}\cdot\underline{p} + i\underline{\sigma}\cdot\underline{r}\times\underline{p}$$
$$= \underline{r}\cdot\underline{p} + i\underline{\sigma}\cdot\underline{L} \quad -(23)$$

where

$$\underline{L} = \underline{r}\times\underline{p} \quad -(24)$$

5) is the classical angular momentum. So the real part of the first term on the right hand side of eq. (23) is:

$$\text{Real}\, H_{11}\phi = \frac{e^2\hbar}{16\pi\epsilon_0 m^2 c^2 r^3}\, \underline{\sigma}\cdot\underline{L}\,\phi \quad -(25)$$

which is the spin orbit coupling in the H atom as usually given.

Note carefully that $\underline{\sigma}\cdot\underline{L}$ in eq. (25) is classical. It is the expectation value:

$$\underline{\sigma}\cdot\underline{L} = \int \phi^* \hat{\underline{\sigma}}\cdot\hat{\underline{L}}\,\phi\, d\tau \quad -(26)$$

where $\wedge$ denotes operator. In quantum mechanics:

$$\hat{\underline{S}} = \frac{\hbar}{2}\hat{\underline{\sigma}} \quad -(27)$$

so

$$\hat{\underline{\sigma}}\cdot\hat{\underline{L}} = \frac{2}{\hbar}\hat{\underline{S}}\cdot\hat{\underline{L}} \quad -(28)$$

In some representations:

$$\hat{\underline{S}}\cdot\hat{\underline{L}}\phi = \frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)\phi \quad -(29)$$

So

$$\hat{\underline{\sigma}}\cdot\hat{\underline{L}}\phi = \hbar\left(j(j+1) - l(l+1) - s(s+1)\right)\phi \quad -(30)$$

6) The classical $\underline{\sigma} \cdot \underline{L}$ of eq. (25) is therefore:

$$\underline{\sigma} \cdot \underline{L} = \langle \hat{\underline{\sigma}} \cdot \hat{\underline{L}} \rangle$$

$$= \hbar \left(j(j+1) - \ell(\ell+1) - s(s+1) \right) \int \phi^* \phi \, d\tau$$

$$= \hbar \left(j(j+1) - \ell(\ell+1) - s(s+1) \right) \quad -(31)$$

From eqs. (25) and (31):

$$\boxed{\text{Real}\, E_{11} = \frac{e^2 \hbar^2}{16\pi \epsilon_0 m^2 c^2} \left(j(j+1) - \ell(\ell+1) - s(s+1) \right) \int \frac{\phi^* \phi}{r^3} \, d\tau} \quad -(32)$$

The second term on the right hand side of eq. (22) is:

$$E_{12}\phi = \frac{i e^2 \hbar}{16\pi \epsilon_0 m^2 c^2} \frac{1}{r} \underline{\sigma} \cdot \underline{\nabla} \phi \, \underline{\sigma} \cdot \underline{p} \quad -(33)$$

It is convenient to develop this with:

$$\underline{\sigma} \cdot \underline{p} = \frac{1}{r^2} \underline{\sigma} \cdot \underline{r} \, \underline{\sigma} \cdot \underline{r} \, \underline{\sigma} \cdot \underline{p} \quad -(34)$$

$$= \frac{1}{r^2} \underline{\sigma} \cdot \underline{r} \left(\underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L} \right)$$

From eqs. (33) and (34) the real part of

) $H_{12}\phi$ can be evaluated from:

$$\underline{\sigma} \cdot \underline{\nabla}\phi\, \underline{\sigma} \cdot \underline{p} = i\, \underline{\sigma} \cdot \underline{\nabla}\phi\, \frac{\underline{\sigma} \cdot \underline{r}\, \underline{\sigma} \cdot \underline{L}}{r^2} + \ldots \qquad -(35)$$

so:

$$\text{Real } H_{12}\phi = -\frac{e^2 \hbar}{16\pi\epsilon_0 m^2 c^2 r^3}\, \underline{\sigma} \cdot \underline{\nabla}\phi\, \underline{\sigma} \cdot \underline{r}\, \underline{\sigma} \cdot \underline{L} \qquad -(36)$$

where:

$$\underline{\sigma} \cdot \underline{\nabla}\phi\, \underline{\sigma} \cdot \underline{r} = \underline{\nabla}\phi \cdot \underline{r} + i\, \underline{\sigma} \cdot \underline{\nabla}\phi \times \underline{r} \qquad -(37)$$

so

$$\text{Real } H_{12}\phi = -\frac{e^2 \hbar}{16\pi\epsilon_0 m^2 c^2 r^3}\, \underline{r} \cdot \underline{\nabla}\phi\, \underline{\sigma} \cdot \underline{L} \qquad -(38)$$

From eqs. (31) and (38):

$$\boxed{H_{12}\phi = -\frac{e^2 \hbar^2}{16\pi\epsilon_0 m^2 c^2 r^3} \left(j(j+1) - l(l+1) - s(s+1) \right) \underline{r} \cdot \underline{\nabla}\phi \qquad -(39)}$$

The energy expectation values are:

$$E_{12} = -\frac{e^2 \hbar^2}{16\pi\epsilon_0 m^2 c^2} \left(j(j+1) - l(l+1) - s(s+1) \right) \int \frac{\phi^* \underline{r} \cdot \underline{\nabla}\phi}{r^3}\, d\tau \qquad -(40)$$

8) The expectation values in eqs. (40) and (32) can be evaluated for the hydrogenic orbitals by computer algebra. In spherical polar coordinates:

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi \quad -(41)$$

and

$$\underline{\nabla}\psi = \frac{\partial\psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\underline{e}_\phi + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\theta}\underline{e}_\theta \quad -(42)$$

Using

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(43)$$

it follows that:

$$\underset{\sim}{r} \cdot \underline{\nabla}\psi = r\frac{\partial\psi}{\partial r} \quad -(44)$$

So:

$$\boxed{E_{12} = -\frac{e^2\hbar^2}{16\pi\epsilon_0 m^2 c^2}\left(j(j+1) - l(l+1) - s(s+1)\right)\int \frac{\psi^*}{r^2}\frac{\partial\psi}{\partial r}\,d\tau}$$

$$-(45)$$

This result is the same as given by the second term of the type II hamiltonian of ~~eq.~~ page 4 of note 252(10), providing a cross check of methods.

9) Incorporation of the Gravitational Field

The gravitational field case is computed in the simplest way using the minimal prescription:

$$E \rightarrow E - e\phi + m\underline{\Phi} \quad -(46)$$

where the gravitational potential is:

$$\underline{\Phi} = -\frac{GM}{r} \quad -(47)$$

Here m is the mass of the electron and M is a mass to which it is attracted. By definition the gravitational force between m and M is:

$$\underline{F} = m\underline{g} = -mMG\frac{\underline{r}}{r^3} \quad -(48)$$

where

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi}. \quad -(49)$$

In analogy:

$$\underline{E} = -\underline{\nabla}\phi \quad -(50)$$

So in the presence of gravitation:

$$\boxed{e\phi \longrightarrow e\phi - m\underline{\Phi}} \quad -(51)$$

in all the preceding equations. So fine structure in atoms and molecules is affected by gravitation. This will be the subject of the next note.