1)

253(1): Some Fundamental Development of the Role of Tetrad in Classical Dynamics and Electrodynamics.

For each quantity in classical dynamics a tetrad can always be defined. For electrodynamics a tetrad can always be defined. For example:

$$e^a = q^a_\mu e^\mu \quad -(1)$$

$$r^a = q^a_\mu r^\mu \quad -(2)$$

$$p^a = q^a_\mu p^\mu \quad -(3)$$

$$a^a = q^a_\mu a^\mu \quad -(4)$$

$$A^a = q^a_\mu A^\mu \quad -(5)$$

where e^a is the unit vector, r^a is the position vector, p^a the linear momentum vector, a^a the linear acceleration vector, A^a the electromagnetic potential vector and so on. Here, a and μ can denote any two bases of the same space of any dimension. Therefore torsion can always be defined.

Cylindrical Polar and Cartesian — (6)

Eq. (1) is:

$$\begin{bmatrix} e^{(1)} \\ e^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}$$

2) where attention is restricted to two dimensions for simplicity of development. Eq. (6) can be written as:

$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_\theta \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(7)$$

linking the unit vectors of the cylindrical polar basis to those of the Cartesian basis:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(8)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(9)$$

so
$$q_\mu^a = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad -(10)$$

which is the rotation matrix about the Z axis. The rotation generator is defined as:

$$J_Z = \frac{1}{i}\frac{d}{d\theta} q_\mu^a \Big|_{\theta=0} \quad -(11)$$

$$= \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and within a factor $\hbar$ is the angular momentum operator of quantum mechanics. This operator therefore originates in the tetrad of Cartan.

3) Now note that:

$$\frac{d}{d\theta} q^a_\mu = \frac{dX}{d\theta}\frac{d}{dX} q^a_\mu \quad -(12)$$

$$= \frac{dY}{d\theta}\frac{d}{dY} q^a_\mu \quad -(13)$$

Here

$$X = r\cos\theta, \quad Y = r\sin\theta \quad -(14)$$

so:

$$\frac{dX}{d\theta} = -r\sin\theta, \quad \frac{dY}{d\theta} = r\cos\theta \quad -(15)$$

Therefore:

$$\frac{dq^a_\mu}{dX} = \frac{d\theta}{dX}\frac{dq^a_\mu}{d\theta} = \frac{d\theta}{dX}\begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad -(16)$$

$$\frac{dq^a_\mu}{dY} = \frac{d\theta}{dY}\frac{dq^a_\mu}{d\theta} = \frac{d\theta}{dY}\begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad -(17)$$

Denoting

$$X = 1, \quad Y = 2 \quad -(18)$$

it follows that:

$$\partial_1 q^{(1)}_2 = \frac{\partial\theta}{\partial X}\cos\theta, \quad \partial_1 q^{(2)}_2 = -\frac{\partial\theta}{\partial X}\sin\theta$$

$$\partial_2 q^{(1)}_1 = -\frac{\partial\theta}{\partial Y}\sin\theta, \quad \partial_2 q^{(2)}_1 = -\frac{\partial\theta}{\partial Y}\cos\theta \quad -(19)$$

Therefore:

$$\sin\theta = -\frac{\partial X}{\partial\theta}\partial_1 q^{(2)}_2 = -\frac{\partial Y}{\partial\theta}\partial_2 q^{(1)}_1 \quad -(20)$$

$$\cos\theta = \frac{\partial X}{\partial\theta}\partial_1 q^{(1)}_2 = -\frac{\partial Y}{\partial\theta}\partial_2 q^{(2)}_1$$

4) Therefore:

$$\sin\theta = r\sin\theta\,\partial_1 q^{(2)}_{\ 2} = -r\cos\theta\,\partial_2 q^{(1)}_{\ 1}$$
$$\cos\theta = -r\sin\theta\,\partial_1 q^{(1)}_{\ 2} = -r\cos\theta\,\partial_2 q^{(2)}_{\ 1}$$
$$\partial_1 q^{(2)}_{\ 2} = \frac{1}{r}, \quad \partial_2 q^{(1)}_{\ 1} = -\frac{1}{r}\tan\theta \qquad -(21)$$
$$\partial_1 q^{(1)}_{\ 2} = -\frac{1}{r}\cot\theta, \quad \partial_2 q^{(2)}_{\ 1} = -\frac{1}{r}$$

By antisymmetry the torsion elements are as follows:

$$T^{(1)}_{\ 12} = 2\left(\partial_1 q^{(1)}_{\ 2} + \omega^{(1)}_{\ 12}\right) = -2\left(\partial_2 q^{(1)}_{\ 1} + \omega^{(1)}_{\ 21}\right)$$
$$= 2\left(-\frac{1}{r}\cot\theta + \omega^{(1)}_{\ 12}\right) = -2\left(-\frac{1}{r}\tan\theta + \omega^{(1)}_{\ 21}\right) \qquad -(22)$$

So:

$$\boxed{\omega^{(1)}_{\ 12} + \omega^{(1)}_{\ 21} = \frac{1}{r}\left(\tan\theta + \cot\theta\right)} \qquad -(23)$$

Also:

$$T^{(2)}_{\ 12} = 2\left(\partial_1 q^{(2)}_{\ 2} + \omega^{(2)}_{\ 12}\right) = -2\left(\partial_2 q^{(2)}_{\ 1} + \omega^{(2)}_{\ 21}\right) \qquad -(24)$$
$$= \frac{1}{r} + \omega^{(2)}_{\ 12} = -\left(-\frac{1}{r} + \omega^{(2)}_{\ 21}\right)$$

so

$$\boxed{\omega^{(2)}_{\ 12} = -\omega^{(2)}_{\ 21}} \qquad -(25)$$

In these equations:

5)

$$\partial_1 q^{(2)}_{2} = \frac{1}{r} = \frac{1}{2}\left(T^{(2)}_{12} - \omega^{(2)}_{12}\right) \quad -(26)$$

$$\partial_2 q^{(2)}_{1} = -\frac{1}{r} = \frac{1}{2}\left(T^{(2)}_{21} - \omega^{(2)}_{21}\right) \quad -(27)$$

It follows that:

$$T^{(2)}_{12} - \omega^{(2)}_{12} = \frac{2}{r} = -\left(T^{(2)}_{21} - \omega^{(2)}_{21}\right) \quad -(28)$$

for these elements.

In general:

$$T^{a}_{\mu\nu} = 2\left(\partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu\nu}\right) \quad -(29)$$

$$= 2\Gamma^{a}_{\mu\nu}.$$

In Cartan geometry:

$$T^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \omega^{a}_{\nu b} q^{b}_{\mu}, \quad -(30)$$

which is the tensor format of the ~~se~~ first Cartan structure equation:

$$T = D \wedge q = d \wedge q + \omega \wedge q \quad -(31)$$

In eq. (30):

$$\omega^{a}_{\mu\nu} = \omega^{a}_{\mu b} q^{b}_{\nu} \quad -(32)$$

$$\omega^{a}_{\nu\mu} = \omega^{a}_{\nu b} q^{b}_{\mu} \quad -(33)$$

°) Therefore:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - (\partial_\nu q^a_\mu + \omega^a_{\nu\mu})$$

$$= \Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu} \quad -(34).$$

The commutator method shows that:

$$\Gamma^a_{\mu\nu} = - \Gamma^a_{\nu\mu} \quad -(35)$$

so

$$T^a_{\mu\nu} = 2(\partial_\mu q^a_\nu + \omega^a_{\mu\nu}) \quad -(36)$$

The above analysis for the cylindrical polar coordinates shows that in general, the spin connection is asymmetric, but the Γ connection is always antisymmetric. The two connections are related by:

$$\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} \quad -(37)$$

In the context of this note we want to relate Cartan geometry in general to the theory of angular momentum, which is basic to quantum mechanics. This method can be illustrated with eq. (10), which defines the fundamental tetrad of the cylindrical polar system:

7)

$$q^a_\mu = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad -(38)$$

so the tetrad is the rotation matrix about Z.

Differentiating eq. (38) with respect to θ gives:

$$\frac{\partial q^a_\mu}{\partial \theta} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad -(39)$$

$$= \frac{\partial x^\mu}{\partial \theta} \partial_\mu q^a_\nu$$

We define the rotation generator matrix by:

$$\boxed{J^a_\mu = \frac{\partial x^\mu}{\partial \theta} \partial_\mu q^a_\nu} \quad -(40)$$

For a Z axis rotation in cylindrical polar coordinates:

$$J^a_\mu = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad -(41)$$

The infinitesimal rotation generator matrix is

$$J^a_\mu := \frac{1}{i} J^a_\mu(\theta = 0) \quad -(42)$$

$$= \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

8) Therefore:

$$\frac{1}{i}\left(\frac{\partial x^{\mu}}{\partial\theta}\partial_{\mu} q^{a}_{\sim}\right)_{\theta=0} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad -(43)$$

Now define the rotation generator matrix in the base manifold:

$$J^{\sim}_{\mu} = q^{\sim}_{a} J^{a}_{\mu} \quad -(44)$$

where the inverse tetrad is defined by:

$$q^{\sim}_{a} q^{a}_{\sim} = 1 \quad -(45)$$

so

$$q^{\sim}_{a} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad -(46)$$

Therefore:

$$J^{\sim}_{\mu} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad -(47)$$

On the other hand, if we define:

$$J^{\sim}_{\mu} = J^{a}_{\mu} q^{\sim}_{a} \quad -(48)$$

then:

$$J^{\sim}_{\mu} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad -(49)$$

9) This checks that both definitions are equivalent.

From eqs. (43) and (47):

$$\boxed{i\, J_\mu^{\nu} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \frac{1}{i}\left(\frac{dx^\mu}{\partial \theta} \partial_\mu q_\nu^a \right)_{\theta=0}}$$

— (50)

Now use

$$J_{\mu\nu} = g_{\nu\rho} J_\mu^\rho \quad -(51)$$

where $g_{\nu\rho}$ is the metric, and denote:

$$J_Z = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad -(51)$$

It is clear that J_Z is the vector format of the tensor $i\, J_\mu^{\nu}$. In a space without curvature in three dimensions:

$$J_i = \epsilon_{ijk} J_{jk} \quad -(52)$$

The angular momentum operator in quantum mechanics is:

$$\boxed{\hat{J}_Z = \hbar J_Z} \quad -(53)$$

so

$$\boxed{\hat{J}_Z = i\hbar\, J_\mu^{\nu} = \frac{\hbar}{i}\left(\frac{dx^\mu}{\partial \theta} \partial_\mu q_\nu^a \right)_{\theta=0}}$$

— (54)

One may define:

$$\overline{J}_{\rho} = \epsilon_{\rho}{}^{\mu}{}_{\nu} J_{\mu}^{\nu} \quad -(55)$$

in any space, where:

$$\epsilon_{\rho}{}^{\mu}{}_{\nu} = g^{\mu\lambda} \epsilon_{\rho\lambda\nu} \quad -(56)$$

so in correct tensor notation eq. (54) is:

$$\boxed{\hat{J}_{\rho} = i\hbar\, \epsilon_{\rho}{}^{\mu}{}_{\nu} J_{\mu}^{\nu}} \quad -(57)$$

The Cartan formalism is very powerful and can be applied to every tetrad. Some examples are given in eqs. (1) to (5). There is also the tetrad defined by the gravitational potential:

$$\underline{\Phi}^{a} = q^{a}_{\mu} \underline{\Phi}^{\mu} \quad -(58)$$

Field equations can be deduced from the Cartan identity:

$$D \wedge T = q \wedge R = R \wedge q \quad -(59)$$

and the Evans identity:

$$D \wedge \tilde{T} = q \wedge \tilde{R} = \tilde{R} \wedge q \quad -(60)$$
