253(6): Relativistic Term in the Fermion Equation

Consider the free particle fermion equation in the format:

$$E = mc^2 + \frac{c^2 p^2}{E + mc^2} \quad -(1)$$

The approximation used in the usual development is:

$$E \sim mc^2 \quad -(2)$$

so

$$E = mc^2 + \frac{c^2 p^2}{2mc^2} \quad -(3)$$

i.e.

$$E - mc^2 = \frac{p^2}{2m} \quad -(4)$$

which produces the non-relativistic kinetic energy:

$$T = E - mc^2 = \frac{p^2}{2m} \quad -(5)$$

More generally:

$$E = \gamma mc^2 \quad -(6)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(7)$$

so

$$E = mc^2 + \frac{c^2 p^2}{mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} + mc^2} \quad -(8)$$

If

$$v << c \quad -(9)$$

2) then:

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{v^2}{2c^2} \quad - (10)$$

In the same approximation (9):

$$p = mv \quad - (11)$$

so

$$E = mc^2 + \frac{1}{m}\frac{p^2}{2 + \frac{p^2}{2m^2c^2}} \quad - (12)$$

where

$$p << mc. \quad - (13)$$

so

$$E - mc^2 = \frac{p^2}{2m}\left(1 + \frac{p^2}{4m^2c^2}\right)$$

$$\sim \frac{p^2}{2m}\left(1 - \frac{p^2}{4m^2c^2}\right) \quad - (14)$$

This quantizes to:

$$H\phi = (E - mc^2)\phi = \frac{p^2}{2m}\phi - \frac{p^4}{8m^3c^2}\phi$$

$$= -\frac{\hbar^2}{2m}\nabla^2\phi + \frac{\hbar^4}{8m^3c^2}\nabla^4\phi \quad - (15)$$

Considering motion of the free fermion in the Z axis:

$$(E - mc^2)\phi = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial Z^2} + \frac{\hbar^4}{8m^3c^2}\frac{\partial^4\phi}{\partial Z^4}$$

$$= -\frac{\hbar^2}{2m}\left(\frac{\partial^2\phi}{\partial Z^2} - \frac{1}{4}\left(\frac{\hbar}{mc}\right)^2\frac{\partial^4\phi}{\partial Z^4}\right) \quad -(16)$$

i.e.

$$\frac{\partial^2\phi}{\partial Z^2} - \frac{1}{4}\left(\frac{\hbar}{mc}\right)^2\frac{\partial^2}{\partial Z^2}\left(\frac{\partial^2\phi}{\partial Z^2}\right) \quad -(17)$$

$$= -\frac{2m}{\hbar^2}(E - mc^2)\phi$$

This is a differential equation of the type:

$$x - \frac{\lambda^2}{4}\frac{\partial^2 x}{\partial Z^2} = A\phi \quad -(18)$$

where

$$x = \frac{\partial^2\phi}{\partial Z^2}, \quad \lambda = \frac{\hbar}{mc} \quad -(19)$$

It gives a relativistic correction to the free particle Schroedinger equation:

$$\frac{\partial^2\phi}{\partial Z^2} = -\frac{2mE}{\hbar^2}\phi \quad -(20)$$

If eq. (17) is solved numerically a new type of quantum tunnelling theory may be developed.