253(5): Evaluation of Terms involving $\underline{\nabla}(\underline{\sigma}\cdot\underline{p})$

The Hamiltonian to be evaluated is the first term in eq. (14) of note 253(2):

$$H\phi = -\frac{ie\hbar}{4m^2c^2}\underline{\nabla}(\underline{\sigma}\cdot\underline{p})\phi \quad -(1)$$

in which $\underline{p}$ is a function. Use:

$$\underline{\sigma}\cdot\underline{p} = \frac{1}{r^2}\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{p} \quad -(2)$$

$$= \frac{1}{r^2}\underline{\sigma}\cdot\underline{r}\left(\underline{r}\cdot\underline{p} + i\underline{\sigma}\cdot\underline{r}\times\underline{p}\right)$$

From eqs. (1) and (2):

$$\mathrm{Re}(H\phi) = \frac{e\hbar}{4m^2c^2}\underline{\nabla}\left(\frac{\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{r}\times\underline{p}}{r^2}\right)\phi \quad -(3)$$

in which:

$$\underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{r}\times\underline{p} = \underline{r}\cdot\underline{r}\times\underline{p} + i\underline{\sigma}\cdot\underline{r}\times(\underline{r}\times\underline{p}) \quad -(4)$$

So

$$\mathrm{Re}(H\phi) = \frac{e\hbar}{4m^2c^2}\underline{\nabla}\left(\frac{\underline{r}\cdot\underline{r}\times\underline{p}}{r^2}\right)\phi = 0 \quad -(5)$$

because

$$\underline{r}\cdot\underline{r}\times\underline{p} = \underline{p}\cdot\underline{r}\times\underline{r} = 0. \quad -(6)$$

So as in note 253(2) the term in $\underline{\nabla}(\underline{\sigma}\cdot\underline{p})$ contributes nothing.

2) However if the $\underline{p}$ is regarded as an operator:

$$\underline{p} = -i\hbar\underline{\nabla} \quad -(7)$$

then:

$$H\phi = -\frac{e\hbar^2}{4m^2c^2}\left(\nabla^2\phi\right)\phi \quad -(8)$$

If:

$$\phi = -\frac{e}{4\pi\epsilon_0 r} \quad -(9)$$

then

$$\nabla^2\phi = -\frac{e}{2\pi\epsilon_0 r^3} \quad -(10)$$

and

$$H\phi = \frac{e^2\hbar^2}{8\pi\epsilon_0 m^2c^2r^3}\phi \quad -(11)$$

and

$$E = \frac{e^2\hbar^2}{8\pi\epsilon_0 m^2c^2}\int\frac{\phi^*\phi}{r^3}\,d\tau \quad -(12)$$

This is an example of the fact that results are different if the second $\underline{p}$ is regarded as a function or an operator.