236(2): Summary of the Problem with Newtonian Dynamics and Introduction to Wehr.

True Newtonian dynamics is defined for an inertial frame, in which there is no spin connection $\underline{\omega}$. So:

$$\underline{v} = \frac{d\underline{r}}{dt}, \quad \underline{a} = \frac{d\underline{v}}{dt}, \quad \underline{F} = m\underline{a} \quad -(1)$$

We denote:

$$\underline{v}_N = \frac{d\underline{r}}{dt}, \quad \underline{a}_N = \frac{d\underline{v}_N}{dt} \quad -(2)$$

and

$$\underline{F}_N = m\,\underline{a}_N. \quad -(3)$$

In the presence of a spin connection the measured velocity is:

$$\underline{v} = \underline{v}_N + \underline{\omega} \times \underline{r} \quad -(4)$$

where the spin connection vector $\underline{\omega}$ is the angular velocity vector.

In Newtonian dynamics:

$$E = \frac{1}{2} m v^2 + U \quad -(5)$$

$$= T + U$$

where T is the kinetic energy and U the potential energy. From eqs. (4) and (5):

$$E = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \omega^2 \right) + U \quad -(6)$$

$$= \frac{1}{2} m \left(v_N^2 + \omega^2 r^2 \right) + U$$

2) In order to obtain conical section orbits:

$$U = -\frac{mMG}{r} \quad -(7)$$

in which there is a force of attraction between m and M:

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(8)$$

This is a Newtonian or central force:

$$\underline{F}_N = m\underline{a}_N = -\frac{mMG}{r^2}\underline{e}_r \quad -(9)$$

Eq. (9) is known as the equivalence principle. The problem is that eq. (9) does not contain the spin connection $\underline{\omega}$.

In Newtonian dynamics eq. (6) is rewritten as:

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V \quad -(10)$$

where the "effective potential" is:

$$V = U + \frac{1}{2}mr^2\omega^2 \quad -(11)$$

$$= U + \frac{L^2}{2mr^2}$$

where L is the angular momentum. The term:

$$U_c = \frac{L^2}{2mr^2} \quad -(12)$$

3) is known as "the centrifugal potential".

The obvious problem is that Π_c is not a potential energy at all, it is part of the kinetic energy as defined in eq. (6). It is part of the kinetic energy defined by the spin connection.

Nevertheless, eq. (5) appears to work very well in such areas as space dynamics and aero-space dynamics. When the spin connection is taken into account the acceleration due to gravity is:

$$\underline{g} = \underline{a}_N + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + 2 \underline{\omega} \times \underline{v}_N + \frac{d\underline{\omega}}{dt} \times \underline{r} \quad -(13)$$

For an elliptical or conical section orbit, the Coriolis terms vanish, so:

$$\underline{g} = \underline{a}_N + \underline{\omega} \times (\underline{\omega} \times \underline{r}) = - \frac{MG}{r^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(14)$$

This is the acceleration due to gravity experienced by a satellite. For the Earth:

$$\omega = 7.29 \times 10^{-5} \text{ rad / sec} \quad -(15)$$

The true Newtonian acceleration is $\underline{a}_N$:

$$\underline{a}_N = \frac{d^2 r}{dt^2} \underline{e}_r \quad -(16)$$

and is directed towards the centre of the earth and is vertical to the earth's surface. However, the experimentally measured acceleration is $\underline{g}$ and is slightly different from the vertical.

From eq. (14), when:

$$\underline{\omega} = \underline{0} \quad -(17)$$

then

$$\underline{g} = \frac{d^2 r}{dt^2} \underline{e}_r = -\frac{MG}{r^2} \underline{e}_r \quad -(18)$$

and when a satellite stops orbiting, i.e. when eq. (17) is true, it falls into the earth.

In previous notes eq. (14) was inferred directly from kinematics, provided that the half right latitude of the ellipse is:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(19)$$

The basic kinematics for planar motion gave the result:

5)

$$\underline{a} = \frac{d^2 r}{dt^2}\,\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= -\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2 \underline{e}_r \quad -(20)$$

and if eq. (19) is used this becomes:

$$\underline{a} = -\frac{MG}{r^2}\,\underline{e}_r \quad -(21)$$

i.e. it becomes the acceleration to the potential:

$$U = -\frac{mMG}{r} \quad -(22)$$

However, it was shown that the acceleration $\underline{a}$ is not an attractive acceleration, it is the equation:

$$\boxed{\frac{d^2 \underline{r}}{dt^2} = -\left(\frac{r}{\alpha}\right)\omega^2 \underline{r}} \quad -(23)$$

For a circle:

$$r = \alpha \quad -(24)$$

so

$$\frac{d^2 \underline{r}}{dt^2} = -\omega^2 \underline{r} \quad -(25)$$

i.e.

$$\boxed{\underline{r} = \underline{r}(0)\exp(i\omega t)} \quad -(26)$$

This is a vector $\underline{r}$ going around in a circle, i.e. a circular orbit.

6) Therefore Newtonian dynamics works with accuracy as an empirical procedure, (eq. (6)), but does not have any explanation of why there is an orbit. It produces the result:

$$\underline{a} = \frac{d^2 r}{dt^2} \underline{e}_r = -\left(\frac{r}{\alpha}\right) \omega^2 r \underline{e}_r$$

$$= -\frac{MG}{r^2} \underline{e}_r \quad -(27)$$

This simply means that a Newtonian procedure produces a Newtonian result.

The true, experimentally observed, result is:

$$\underline{g} = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}), \quad -(28)$$

and this is also the kinematic result. For an elliptical orbit:

$$\boxed{\underline{g} = \left(\frac{L}{mr}\right)^2 \left(\frac{1}{r} - \frac{1}{\alpha}\right) \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(29)}$$

The ECE explanation is that the orbit exists because of the spin connection $\underline{\omega}$, which produces the kinematic eq. (23).

7) G. Wehr has also produced a criticism of the Newtonian dynamics. He bases his procedure on the Frenet unit vectors. These are defined in: "Vector Analysis Problem Solver", problem 5.2 ff.

Let C be a curve in space defined by:

$$\underset{\sim}{r} = \underset{\sim}{r}(s) \quad -(30)$$

the unit tangent vector is:

$$\underline{T} = \frac{dr}{ds} \quad -(31)$$

the Frenet vectors are defined by:

$$\frac{d\underline{T}}{ds} = \frac{1}{\rho}\underline{N} \quad -(32)$$

$$\frac{d\underline{B}}{ds} = -\tau\underline{N} \quad -(33)$$

$$\frac{d\underline{N}}{ds} = -\frac{1}{\rho}\underset{\sim}{T} + \tau\underline{B} \quad -(34)$$

where $\underline{N}$ is the principal normal and $\underline{B}$ is the binormal.

$$\underline{B} = \underline{T} \times \underline{N} \quad -(35)$$

Here ρ is the radius of curvature. The quantity τ is the Frenet torsion.

8) Now let a curve in space be defined by

$$\underline{r} = \underline{r}(t) \quad -(36)$$

(VAPS problem 5-15), then:

$$\frac{d\underline{r}}{dt} = \frac{ds}{dt}\underline{T} \quad -(37)$$

and

$$\frac{d^2\underline{r}}{dt^2} = \frac{ds}{dt}\frac{d\underline{T}}{dt} + \frac{d^2 s}{dt^2}\underline{T} \quad -(38)$$

On page 22 of Weber he uses the parameterization:

$$ds = |d\underline{r}| \quad -(39)$$

and defines the velocity as:

$$\underline{v} = \frac{d\underline{r}}{dt} = v\underline{T} \quad -(40)$$

where

$$v = \frac{ds}{dt} \quad -(41)$$

He defines the acceleration $\underline{b}$ in his eq. (1.53) by eq. (38) i.e.

$$\underline{b} = \frac{d\underline{v}}{dt} = \frac{dv}{dt}\underline{T} + v\frac{d\underline{T}}{dt} \quad -(42)$$

In his eq. (1.54) he defines the angle ϕ by:

$$\frac{d\underline{T}}{d\phi} = \underline{N} \quad -(43)$$

9) Comparing eqs. (32) and (43):

$$\underline{N} = \rho \frac{d\underline{T}}{ds} = \frac{d\underline{T}}{d\phi} \quad -(44)$$

so
$$ds = \rho \, d\phi \quad -(45)$$

as in Wehr's eq. (1.58), where he writes:

$$ds = \rho \, d\phi, \quad \frac{ds}{dt} = \rho \frac{d\phi}{dt}, \quad \frac{d\phi}{dt} = \frac{v}{\rho} \quad -(46)$$

He defines the angular velocity as:

$$\omega = \frac{d\phi}{dt} \quad -(47)$$

so:
$$v = \omega \rho \quad -(48)$$

Note that for the plane polar coordinates:

$$\underline{r} = r\cos\theta \, \underline{i} + r\sin\theta \, \underline{j} \quad -(49)$$

so
$$d\underline{r} = \frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \theta} d\theta \quad -(50)$$

$$= (\cos\theta \, dr - r\sin\theta \, d\theta)\, \underline{i} + (\sin\theta \, dr + r\cos\theta \, d\theta)\, \underline{j}$$

so
$$ds^2 = d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2 \quad -(51)$$

so eq. (39) used by Wehr assumes that

10) From eq. (50):

$$|d\underline{r}| = \left((\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 \right)^{1/2} \quad -(52)$$

$$= \left(dr^2 + r^2 d\theta^2 \right)^{1/2}$$

So Wehr uses:

$$\boxed{ds = |d\underline{r}| = \left(dr^2 + r^2 d\theta^2 \right)^{1/2}} \quad -(53)$$

Wehr then uses the procedure:

$$\underline{N} = \frac{d\underset{\sim}{T}}{d\phi} = \rho \frac{d\underset{\sim}{T}}{ds}, \quad -(54)$$

so

$$\underline{b} = \frac{dv}{dt} \underset{\sim}{T} + v \frac{d\phi}{dt} \underline{N}$$

$$\underline{b} = \frac{dv}{dt} \underset{\sim}{T} + \omega v \underline{N} \quad -(55)$$

In plane polar coordinates this is equivalent to:

$$\underline{a} = \frac{dv}{dt} \underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \underline{e}_\theta \quad -(56)$$

In Wehr's notation:

$$\underline{b} = \underline{b}_T + \underline{b}_N = \frac{dv}{dt} \underset{\sim}{T} + \frac{v^2}{\rho} \underline{N} \quad -(57)$$

1) The plane polar representation is given by Webster's eqn. (2.8):

$$\underline{a} = \frac{d^2 \underline{r}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e}_\theta \quad -(58)$$

Webster compares eqns. (57) and (58) using:

$$r\dot{\theta}^2 = \rho\dot{\phi}^2 = \frac{v^2}{\rho} \quad -(59)$$

and

$$\underline{e}_r = -\underline{N} \quad -(60)$$

He writes:

$$\underline{a} = \underline{b}_T + \underline{b}_0 + b_{\ddot{\rho}} \quad -(61)$$

$$= \frac{dv}{dt}\underset{\sim}{T} + \left(\frac{v^2}{\rho} - \ddot{\rho}\right)\underline{N}$$

and uses the assumptions

$$r = \rho \quad -(62)$$

and

$$\underline{e}_\theta = \underline{T} \quad -(63)$$

It follows that

$$\boxed{\frac{dv}{dt} = 2\dot{r}\dot{\theta} + r\ddot{\theta}} \quad -(64)$$

and

$$\boxed{\ddot{r} - r\dot{\theta}^2 = \ddot{\rho} - \frac{v^2}{\rho}} \quad -(65)$$

12) The quantity:

$$\underline{b}_{\ddot{\rho}} = -\ddot{\rho}\,\underline{N} = \ddot{r}\,\underline{e}_r \quad -(66)$$

is called "Krümmungbeschleunigung"

Therefore it seems that Wehr is using an essentially identical notation. The quantity (66) is the true Newtonian acceleration. The Frenet geometry is a special case of Cartan geometry.

The analysis by Wehr is the same as:

VAPS Problem 5-26, page 188

If a particle moves in the XY plane with velocity $\underline{v}$, its acceleration is:

$$\boxed{\underline{a} = \frac{dv}{dt}\,\underline{T} + \frac{v^2}{\rho}\,\underline{N}} \quad -(67)$$

The unit tangent vector is defined as:

$$\underline{T} = \underline{v}/v \quad -(68)$$

$$= \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$$

$$= \underline{e}_r$$

This is defined opposite to Wehr.

13)

Then:

$$\frac{d\underline{v}}{dt} = \frac{d}{dt}\left(v\underline{T}\right) = v\frac{d\underline{T}}{dt} + \frac{dv}{dt}\underline{T} \quad -(69)$$

From eq. (68):

$$\frac{d\underline{T}}{dt} = \frac{d\theta}{dt}\left(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right) = \underline{n} \quad -(70)$$

The vector $\underline{n}$ is perpendicular to $\underline{T}$ and has magnitude:

$$\omega = \left|\frac{d\theta}{dt}\right| \quad -(71)$$

So

$$\underline{n} = \omega\,\underline{N} \quad -(72)$$

If

$$\theta = \theta(s) \quad -(73)$$

then:

$$\left|\frac{d\theta}{dt}\right| = \left|\frac{d\theta}{ds}\right|\frac{ds}{dt} = \frac{1}{\rho}v \quad -(74)$$

where

$$\rho = \left|\frac{ds}{d\theta}\right|, \quad -(75)$$

is the radius of curvature.

So:

$$\underline{n} = \frac{v}{\rho}\underline{N} \quad -(76)$$

14) Therefore:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{v^2}{\rho} \underline{N} + \frac{dv}{dt} \underline{T} \quad -(77)$$

Note finally that:

$$\underline{n} = \frac{d\theta}{dt} \underline{e}_\theta \quad -(78)$$

where

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(79)$$

so

$$\boxed{\begin{aligned} \underline{N} &= \underline{e}_\theta \\ \underline{T} &= \underline{e}_r \end{aligned}} \quad \begin{aligned} &-(80) \\ &-(81) \end{aligned}$$

Wehr and the other hand defines:

$$\underline{N} = -\underline{e}_r \quad -(82)$$

$$\underline{T} = \underline{e}_\theta \quad -(83)$$

Wehr's work is a straightforward adaptation of recent AIAS work.