236(3): Calculation of the Conical Section from the Inverse Square Law of Robert Hooke and Isaac Newton.

The starting equation is:

$$\underline{a} = \frac{d\underline{v}}{dt} = -\frac{k\,\underline{r}}{r^3} \quad -(1)$$

which assumes that the acceleration $\underline{a}$ is inversely proportional to the square of r. At this point in the calculation there is no mention of mass m, mass M and G. Eq. (1) is a pure equation of motion. Note that:

$$\underline{r} \times \frac{d\underline{v}}{dt} = -k\,\underline{r} \times \frac{\underline{r}}{r^3} = \underline{0}. \quad -(2)$$

Also:

$$\frac{d}{dt}(\underline{r} \times \underline{v}) = \frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times \frac{d\underline{v}}{dt} \quad -(3)$$

$$= \underline{v} \times \underline{v} + \underline{r} \times \frac{d\underline{v}}{dt} = \underline{0}.$$

Therefore

$$\underline{r} \times \underline{v} = \text{constant} := \underline{d}. \quad -(4)$$

Denote:

$$\underline{d} = \underline{r} \times \underline{v} = \underline{r} \times \frac{d\underline{r}}{dt} = r\,\underline{r_0} + \frac{d}{dt}(r\,\underline{r_0}) \quad -(5)$$

where:

$$\underline{r_0} := \frac{\underline{r}}{r}. \quad -(6)$$

Using the Leibnitz theorem:

$$\frac{d}{dt}(r\,\underline{r_0}) = \frac{dr}{dt}\,\underline{r_0} + r\,\frac{d\underline{r_0}}{dt} \quad -(7)$$

So:

$$\underline{d} = r\,\underline{r_0} \times \left(\frac{dr}{dt}\,\underline{r_0} + r\,\frac{d\underline{r_0}}{dt}\right) \quad -(8)$$

2) Now use the vector equation:

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b}) \quad -(9)$$

to evaluate:

$$\frac{d\underline{v}}{dt} \times \underline{d} = -\frac{k\underline{r_0}}{r^2} \times \underline{d} \quad -(10)$$

$$= -\frac{k\underline{r_0}}{r^2} \times \left(\underline{r} \times \frac{d\underline{r}}{dt}\right)$$

$$= \underline{r}\left(-\frac{k\underline{r_0}}{r^2} \cdot \frac{d\underline{r}}{dt}\right) - \frac{d\underline{r}}{dt}\left(-\frac{k\underline{r_0}}{r^2} \cdot \underline{r}\right)$$

$$= \frac{d\underline{r}}{dt}\left(\frac{k\underline{r_0}}{r^2} \cdot r\underline{r_0}\right) - \frac{k}{r^2}\left(\underline{r_0} \cdot \frac{d\underline{r}}{dt}\right)\underline{r},$$

where $\underline{r_0} = \frac{\underline{r}}{r}$. $-(11)$

Now note that:

$$\underline{r_0} \cdot \frac{d\underline{r_0}}{dt} = \frac{1}{2}\frac{d}{dt}(\underline{r_0} \cdot \underline{r_0}) = 0. \quad -(12)$$

therefore:

$$\frac{d\underline{v}}{dt} \times \underline{d} = k\frac{d\underline{r}}{dt}\left(\frac{\underline{r_0} \cdot r\underline{r_0}}{r^2}\right) \quad -(13)$$

$$= k\frac{d\underline{r}}{dt}\left(\frac{r_0^2}{r}\right) = k\frac{d\underline{r_0}}{dt}$$

where $\underline{r_0} = \frac{\underline{r}}{r}$ $\quad -(14)$

Therefore:

3)

$$\frac{d}{dt}\left(\underline{v} \times \underline{d}\right) = k \frac{d\underline{r_0}}{dt} \quad \text{— (15)}$$

Therefore: $$\underline{v} \times \underline{d} = \int k \frac{d\underline{r_0}}{dt}\, dt \quad \text{— (16)}$$

$$= k\underline{r_0} + \underline{P}$$

where $\underline{P}$ is a constant of integration.

It follows ~~that~~:

$$\left(\underline{v} \times \underline{d}\right) \cdot \underline{r} = kr + \underline{r} \cdot \underline{P} \quad \text{— (17)}$$

Now express: $$\underline{r} \cdot \underline{P} = rP\cos\theta \quad \text{— (18)}$$

and use:

$$\left(\underline{a} \times \underline{b}\right) \cdot \underline{c} = \left(\underline{b} \times \underline{c}\right) \cdot \underline{a} = \left(\underline{c} \times \underline{a}\right) \cdot \underline{b} \quad \text{— (19)}$$

So: $$\left(\underline{v} \times \underline{d}\right) \cdot \underline{r} = \underline{d} \cdot \left(\underline{r} \times \underline{v}\right) = \underline{d} \cdot \underline{d} = d^2 \quad \text{— (20)}$$

therefore eq. (17) is:

$$d^2 = kr + rP\cos\theta \quad \text{— (21)}$$

i.e. $$\boxed{r = \frac{d^2 / k}{1 + \frac{P}{k}\cos\theta}} \quad \text{— (22)}$$

This is the equation of a conical section.

It has been proven that the solution of:

$$\boxed{\frac{d^2 \underline{r}}{dt^2} = -k \frac{\underline{r}}{r^3}} \quad -(23)$$

is

$$r = \frac{\alpha}{1 + \epsilon \cos\theta}, \quad -(24)$$

where

$$\alpha = \frac{d^2}{k}, \quad \epsilon = \frac{p}{k}. \quad -(25)$$

Now note that the kinematic result of eq. (20) of note 236(2) is:

$$\frac{d^2 \underline{r}}{dt^2} = -\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2 \underline{e}_r \quad -(26)$$

where

$$\underline{e}_r = \frac{\underline{r}}{r} \quad -(27)$$

so

$$\boxed{\frac{d^2 \underline{r}}{dt^2} = -\frac{L^2}{\alpha m^2} \frac{\underline{r}}{r^3}} \quad -(28)$$

From eqs. (23) and (28) it follows that:

$$\boxed{k = \frac{L^2}{\alpha m^2}} \quad -(29)$$

5) Note that if the usual Newtonian theory is used:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(30)$$

so

$$\boxed{k = MG} \quad -(31)$$

and eq. (23) becomes:

$$\underline{a} = \frac{d^2 \underline{r}}{dt^2} = -MG\frac{\underline{r}}{r^3} \quad -(32)$$

i.e.

$$\underline{F} = m\underline{a} = -\frac{mMG}{r^2}\underline{e}_r \quad -(33)$$

which is the principle of equivalence.

By comparing eqs. (22) and (24):

$$\alpha = \frac{d^2}{k} = \frac{d^2}{MG} \quad -(34)$$

so

$$d^2 = \alpha MG = \frac{L^2}{m^2} \quad -(35)$$

i.e.

$$\boxed{d = L/m} \quad -(36)$$

These results are discussed as follows in the context of recent papers.

1) The most important result is that the entire Newtonian theory is the trajectory produced by an equation of motion, eq. (23) or eq. (28). The equation can be expressed as:

$$\frac{d^2 \underline{r}}{dt^2} = \frac{r}{\alpha} \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(37)$$

for an elliptical trajectory (eq. (142) of UFT 235). So the Newtonian theory is simply the description of an ellipse, or elliptical curve. It means that if:

$$\underline{r} = r \, \underline{e}_r \quad -(38)$$

where

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(39)$$

then

$$\frac{d^2 \underline{r}}{dt^2} = - k \frac{\underline{r}}{r^3} \quad -(40)$$

where

$$k = MG \quad -(41)$$

2) The theory is produced by a choice of integration, i.e. $\underline{f}$ in eq. (16) is assumed to be non-zero. If $\underline{f}$ is zero, then:

$$k \, \underline{r}_0 = \underline{v} \times \underline{d} \quad -(42)$$

in eq. (16), and from eq. (22)

$$r = \alpha = \frac{d^2}{k} = \text{constant} \quad -(43)$$

) If $\underline{r}$ is a constant, then there is no acceleration $\underline{a}$, and no force.

3) The Newtonian theory assumes that $\underline{r}$ is not parallel to $\underline{v}$, so:

$$\underline{d} = \frac{\underline{L}}{m} = \underline{r} \times \underline{v} = \text{constant} \quad -(44)$$

so the theory assumes the existence of constant angular momentum:

$$\underline{L} = \underline{r} \times \underline{p} = \text{constant} \quad -(45)$$

If $\underline{r} \parallel \underline{v}$ then:

$$\underline{L} = \underline{0} \quad -(46)$$

and from eq. (15):

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{0} \quad -(47)$$

and we arrive again at:

$$\underline{r} = \text{constant} \quad -(48)$$

Therefore the theory cannot be interpreted simply as M attracting m along $\underline{r}$, because in this case $\underline{r}$ would change.

In order to see why $\underline{v}$ is not parallel to $\underline{r}$ plane polar coordinates are needed. These give:

$$\underline{r} = r\,\underline{e}_r \quad -(49)$$

$$\underline{v} = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(50)$$

$$= \frac{dr}{dt}\underline{e}_r + \omega r\,\underline{e}_\theta$$

and automatically introduce the spin connection:

$$\underline{\omega} = \omega\,\underline{k} \quad -(51)$$

so

$$\underline{v} = \frac{dr}{dt}\underline{e}_r + \underline{\omega}\times\underline{r} \quad -(52)$$

and

$$\underline{v} \not\parallel \underline{r}. \quad -(53)$$

So the real explanation of orbits is $\underline{\omega}$. This is true for all orbits, not just the conical sections.

The kinematics of plane polar coordinates give:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(54)$$

without assuming any kind of force law.

In eq. (54):

$$(\ddot{r} - r\dot{\theta}^2)\underline{e}_r = \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega}\times(\underline{\omega}\times\underline{r}) \quad -(55)$$

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta = \frac{d\underline{\omega}}{dt}\times\underline{r} + 2\underline{\omega}\times\underline{\dot{r}} \quad -(56)$$

9) Here:

$$\underline{L} = mr^2\underline{\omega} = mr^2\omega\underline{k} \quad -(57)$$

The kinematics show that the origin of $\underline{L}$ is the spin connection $\underline{\omega}$, due to the rotation of axes, i.e. the rotation of space itself, a relativistic concept.

This concept of rotating space is missing completely from Newtonian dynamics. The latter assumes the orbit, and does not explain it.

In previous work it has been shown that for all planar orbits, the Coriolis acceleration (56) vanishes. So for all planar orbits:

$$\boxed{\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})} \quad -(58)$$

So eq. (1) is replaced by the more general:

$$* \quad \boxed{\left(\frac{d^2 r}{dt^2}\right)\underline{e}_r = \underline{a} - \underline{\omega} \times (\underline{\omega} \times \underline{r})} \quad -(59)$$

for any planar orbit:

$$r = f(\theta). \quad -(60)$$

From eq. (37) of UFT 234:

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \left(\frac{dr}{d\theta}\right) \frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right), \quad -(61)$$

so for all planar orbits:

$$\left(\frac{L}{mr^2}\right)\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)\underline{e}_r = \underline{a} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(62)$$

with:

$$\underline{L} = mr^2\underline{\omega} \quad -(63)$$

Therefore:

$$\boxed{\underline{a} = \frac{d^2\underline{r}}{dt^2} = \left(\frac{L}{mr^2}\right)\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})} \quad -(64)$$

Here

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} \quad -(65)$$

Eq. (64) generalizes eq. (1) and makes eq. (1) into a covariantly relativistic theory.

In eq. (64): $\quad -(66)$

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r\,\underline{e}_r = -\frac{L^2}{m^2 r^3}\underline{e}_r$$

11) so eq. (64) becomes:

$$\underline{a} = \left(\frac{L}{mr^2}\right)\left[\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{L}{mr}\right]\underline{e}_r \quad -(67)$$

which is true for all planar orbits, not just an elliptical orbit.

Special Case of ~~G~~ Elliptical Orbit.

In this case:

$$\frac{d^2 r}{dt^2}\underline{e}_r = \left(\frac{L}{mr}\right)^2\left(\frac{1}{r} - \frac{1}{\alpha}\right) \quad -(68)$$

(eq. (129) of UFT 235), so:

$$\underline{a} = \left(\frac{L}{mr}\right)^2\left(\frac{1}{r} - \frac{1}{\alpha} - \frac{1}{r}\right)\underline{e}_r$$

$$= -\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2\underline{e}_r. \quad -(69)$$

Using:

$$\underline{e}_r = \frac{\underline{r}}{r} \quad -(70)$$

eq. (69) is

$$\underline{a} = -\frac{L^2}{\alpha m^2}\left(\frac{\underline{r}}{r^3}\right) \quad -(71)$$

which is eq. (28), QED.

The Newtonian theory is very successful as

12) an empirical theory because it uses eq. (1) in the form of the equivalence principle:

$$\underline{F} = m\,\underline{a} = -\frac{mMG}{r^2}\underline{e}_r \quad -(72)$$

and introduces the concept of total energy E, kinetic energy T and potential energy U, where:

$$E = T + U. \quad -(73)$$

It also introduces the concept:

$$F = -\frac{\partial U}{\partial r} = ma \quad -(74)$$

and the concept of work and work integral. However, in order to calculate the ellipse from eq. (73), eq. (50) is used, i.e. plane polar coordinates are used. So:

$$E = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{mMG}{r}. \quad -(75)$$

This is eq. (24) provided that:

$$\alpha = \frac{L^2}{m^2 MG}, \quad \epsilon = \left(1 + \frac{2EL^2}{m^3 M^2 G^2}\right)^{1/2} \quad -(76)$$

So eq. (75) is simply the ellipse itself. The only thing that the Newton theory does is to describe an ellipse, it does not explain why the ellipse exists. ECE explains why an ellipse exists.