236(1): A Comparison of Kinematic and Newtonian Dynamics

Consider the linear velocity $\underline{v}$ in plane polar coordinate:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\underline{e}_r\right)$$

$$= \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(1)$$

$$= \frac{dr}{dt}\underline{e}_r + \omega r\underline{e}_\theta$$

$$= \frac{dr}{dt}\underline{e}_r + \underline{\omega}\times\underline{r}.$$

Here

$$\underline{\omega} = \frac{d\theta}{dt}\underline{k} \quad -(2)$$

is the spin connection and angular velocity vector.

Therefore:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2 \quad -(3)$$

$$= \left(\frac{dr}{dt}\right)^2 + \underline{\omega}\times\underline{r}\cdot\underline{\omega}\times\underline{r}$$

where:

$$\underline{\omega}\times\underline{r}\cdot\underline{\omega}\times\underline{r} = (\underline{\omega}\cdot\underline{\omega})(\underline{r}\cdot\underline{r}) - (\underline{\omega}\cdot\underline{r})^2$$

$$= \omega^2 r^2. \quad -(4)$$

The angular momentum is:

$$\underline{L} = \underline{r}\times\underline{p} \quad -(5)$$

, i.e.

$$\underline{L} = m\,\underline{r} \times \underline{v}$$

$$= m r\,\underline{e}_r \times \left(\frac{dr}{dt}\,\underline{e}_r + \omega r\,\underline{e}_\theta \right)$$

$$= m r^2 \omega\,\underline{e}_r \times \underline{e}_\theta \qquad -(6)$$

$$= m r^2 \omega\,\underline{k}\,.$$

Therefore by <u>pure</u> <u>kinematics</u>:

$$L = m r^2 \omega \qquad -(7)$$

From eq. (7) in eq. (3):

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2} \qquad -(8)$$

and:

$$\boxed{\underline{\omega} \times \underline{r} \cdot \underline{\omega} \times \underline{r} = r^2 \left(\frac{d\theta}{dt}\right)^2 = \frac{L^2}{m^2 r^2}} \qquad -(9)$$

<u>This</u> is <u>a result</u> of <u>pure</u> <u>kinematics</u>.

It can be expressed as:

$$\boxed{\left(\frac{dr}{dt}\right)^2 = v^2 - \frac{L^2}{m^2 r^2}} \qquad -(10)$$

3) Newtonian dynamics asserts that:

$$v^2 = \frac{2}{m}(E - U) \quad -(11)$$

and from the point of view of pure kinematics splits v^2 into a combination of the total energy E and the potential energy U. From eq. (11):

$$E = \frac{1}{2}mv^2 + U \quad -(12)$$

Newtonian dynamics defines the kinetic energy as:

$$T = \frac{1}{2}mv^2 \quad -(13)$$

Therefore in Newtonian physics:

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m}(E - U) - \frac{L^2}{m^2 r^2} \quad -(14)$$

It adds no new knowledge and makes the kinematics more complicated by introducing E, T and U.

The kinematic acceleration is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta . \quad -(15)$$

As shown in previous notes the Coriolis acceleration vanishes for all planar orbits, so:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r \qquad -(16)$$

$$= \left(\frac{d^2 r\,\mathbf{e}_r}{dt^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right)$$

for all planar orbits, in which:

$$\mathbf{v} = \frac{dr}{dt}\,\mathbf{e}_r + \boldsymbol{\omega} \times \mathbf{r}. \qquad -(17)$$

So:

$$\mathbf{a} = \frac{d}{dt}\left(\frac{dr}{dt}\right)\mathbf{e}_r + \boldsymbol{\omega} \times \mathbf{v}_o$$

$$\boxed{\mathbf{a} = \frac{d\mathbf{v}_s}{dt} + \boldsymbol{\omega} \times \mathbf{v}_o} \qquad -(18)$$

where $\mathbf{v}_s$ is:

$$\mathbf{v}_s = \frac{dr}{dt}\,\mathbf{e}_r \qquad -(19)$$

and where $\mathbf{v}_o$ is:

$$\mathbf{v}_o = \boldsymbol{\omega} \times \mathbf{r}. \qquad -(20)$$

These are all results of pure kinematics.

As in previous notes and papers consider the ellipse and elliptical or conical section orbit:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \qquad -(21)$$

), then:

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \left(\frac{1}{r} - \frac{1}{\alpha}\right) \quad -(22)$$

for this type of orbit.

From eq. (10) for the general orbit:

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(v^2 - \frac{L^2}{m^2 r^2}\right)^{1/2}$$

$$= \frac{dr}{dt}\frac{d}{dr}\left(v^2 - \frac{L^2}{m^2 r^2}\right)^{1/2} \quad -(23)$$

Therefore for any orbit in a plane, pure kinematics give:

$$\boxed{\frac{d^2 r}{dt^2} = \frac{1}{2}\frac{d}{dr}\left(v^2 - \frac{L^2}{m^2 r^2}\right)} \quad -(24)$$

In eq. (24):

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \omega^2 r^2$$

$$= \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2} \quad -(25)$$

Using eq. (25) in eq. (24):

6)

$$\frac{d^2 r}{dt^2} = \frac{1}{2}\frac{d}{dr}\left(\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2} - \frac{L^2}{m^2 r^2}\right)$$

$$= \frac{1}{2}\frac{d}{dr}\left(\frac{dr}{dt}\right)^2$$

$$= \frac{d^2 r}{dt^2} \qquad -(26)$$

Therefore eq. (24) is an identity:

$$\boxed{\frac{d^2 r}{dt^2} := \frac{1}{2}\frac{d}{dr}\left(v^2 - \left(\frac{L}{mr}\right)^2\right)} \qquad -(27)$$

in which := means "identically equal to". Eq. (27) is a pure kinematic result which is true for all planar orbits. Note carefully that eq. (27) relies neither on Newtonian nor Lagrangian theory. It is a perfectly general result for all planar orbits.

For the conical section orbits eqs. (22) and (24) show that:

).

$$\boxed{\frac{1}{2}\frac{dv^2}{dr} = -\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2} \quad -(28)$$

Newtonian dynamics is less general, and asserts that:

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{2}{m}\left(E-U\right) - \omega^2 r^2\right)^{1/2}$$

$$= \frac{dr}{dt}\frac{d}{dr}\left(\frac{2}{m}\left(E-U\right) - \frac{L^2}{m^2r^2}\right)^{1/2}$$

$$= \frac{1}{2}\frac{d}{dr}\left(\frac{2}{m}\left(E-U\right) - \frac{L^2}{m^2r^2}\right) \quad -(29)$$

In Newtonian dynamics, E is a constant of motion:

$$\frac{dE}{dt} = 0, \quad -(30)$$

So:

$$\frac{dE}{dr} = \frac{dE}{dt}\frac{dt}{dr} = 0. \quad -(31)$$

Therefore in Newtonian dynamics:

$$\frac{d^2r}{dt^2} = \frac{1}{2}\left(-\frac{2}{m}\frac{dU}{dr} + \frac{2L^2}{m^2r^3}\right) \quad -(32)$$

i.e.

$$\left(\frac{d^2r}{dt^2}\right)_{Newton} = -\frac{1}{m}\frac{dU}{dr} + \frac{L^2}{m^2r^3} \quad -(33)$$

Comparison of the identity (27) with eq. (33) shows that Newtonian dynamics assume:

$$\boxed{\frac{m}{2}\frac{dv^2}{dr} = -\frac{dU}{dr}} \quad -(34)$$

It is further assumed in Newtonian dynamics that

$$F = -\frac{dU}{dr} \quad -(35)$$

So from eqs. (34) and (35):

$$F = m\frac{dv}{dt} \quad -(36)$$

Finally Newtonian dynamics assume ~~that~~:

$$U = -\frac{mMG}{r} \quad -(37)$$

in which an object of mass m is attracted to an object of mass M. Here G is Newton's constant. It is claimed in Newtonian dynamics

) that eqs. (34) to (37) produce a conical section orbit. However for this claim to be true, eqs. (28) and (34) mean that:

$$-\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2 = -\frac{mMG}{r^2} \qquad -(38)$$

So:

$$\alpha = \frac{L^2}{m^2 MG} \qquad -(39)$$

The Newtonian theory produces the conical sections if and only if eq. (39) is true. There is no reason why eq. (37) should be true. The only thing that is observed is eq. (28).

The Newtonian theory produces:

$$\boxed{\left(\frac{d^2 r}{dt^2}\right)_{\text{Newton}} = -\frac{MG}{r^2} + \frac{L^2}{m^2 r^3}} \qquad -(40)$$

This is always misinterpreted as a force:

$$\underline{F} = \left(-\frac{mMG}{r^2} + \frac{L^2}{mr^3}\right)\underline{e}_r \qquad -(41)$$

i.e. as a negative force of attraction balanced by a positive "pseudoforce" of repulsion. From eq. (9) it is clear that the "pseudoforce" is part of the kinetic energy, and comes from the orbital velocity. It has nothing to do with potential energy...

Newtonian dynamics fails completely when consideration is taken of eq. (16), which shows that the total acceleration is:

$$\underline{a} = \frac{d^2 r \underline{e}_r}{dt^2} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= \left(\frac{d^2 r}{dt^2} - \omega^2 r \right) \underline{e}_r \quad -(42)$$

$$\boxed{\underline{a} = \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r} \quad -(43)$$

In eq. (43):

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \underline{e}_r = -\frac{L^2}{m^2 r^3} \underline{e}_r \quad -(44)$$

11) From a comparison of eqs. (40) and (44), it is seen very clearly that Newtonian dynamics misses the term $\underline{\omega} \times (\underline{\omega} \times \underline{r})$ of pure kinematics, i.e. Newtonian dynamics uses:

$$\underline{a} = \frac{d^2 r}{dt^2} \underline{e}_r \quad - (45)$$

but the correct kinematic result is:

$$\underline{a} = \left(\frac{d^2 r}{dt^2} - r\dot{\theta}^2 \right) \underline{e}_r \quad - (46)$$

$$= \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r$$

The correct explanation of orbits is that they are generated by a spinning frame of reference which is described by the Cartan connection $\underline{\omega}$ known as angular velocity.

In the next note another criticism of Newtonian dynamics will be discussed, by:
G. Wehr, "Unsolved Problems of Physics"
(Haag and Herchen 1999)