236(4): Self Consistency with Lagrangian Dynamics

From previous notes, for any planar orbit:

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r \qquad -(1)$$

$$= \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= \left(\frac{L}{mr}\right)^2 \left[\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r}\right\}\underline{e}_r$$

In this equation:

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = \frac{d\theta}{dr}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) \qquad -(2).$$

So

$$\underline{a} = \left(\frac{L}{mr}\right)^2\left[\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r}\right\}\underline{e}_r. \qquad -(3)$$

Now note that:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{d\theta} \qquad -(4)$$

$$= -\frac{1}{r^2}\frac{dr}{d\theta} \qquad -(5)$$

So:

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = -\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right) = -\frac{d^2 r}{d\theta^2}$$

2)

So:

$$\underline{a} = -\left(\frac{L}{mr}\right)^2 \left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \underline{e}_r \quad -(6)$$

i.e.

$$\boxed{\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \underline{e}_r = -\left(\frac{mr}{L}\right)^2 \underline{a}} \quad -(7)$$

Denote:

$$\underline{F} = m\,\underline{a} \quad -(8)$$

then eq. (7) becomes:

$$\boxed{\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \underline{e}_r = -\frac{mr^2}{L^2}\underline{F}} \quad -(9)$$

This is the same as equation (7.21) of Marion and Thornton page 250 of the third edition.

Eq. (9) can be obtained from the lagrangian

$$\mathcal{L} = \frac{1}{2} m \left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - U \quad -(10)$$

where

$$U = -\frac{mMG}{r} \quad -(11)$$

) and the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(12)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(13)$$

The angular momentum is:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \frac{d\theta}{dt} = \text{constant} \quad -(14)$$

Eq. (9) has been used extensively in previous work on the general planar orbit, and it has become clear that its origin is pure kinematics. This confirms our finding that for all planar orbits the Coriolis acceleration vanishes:

$$\underline{a}_{\text{Coriolis}}(\text{all orbits}) = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

$$= \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\,\underline{\omega} \times \dot{\underline{r}} \quad -(15)$$

$$= \underline{0}$$

* $\boxed{\text{Eq. (9) contains the centrifugal acceleration } \underline{\omega} \times (\underline{\omega} \times \underline{r}).}$

4) Elliptical Orbit

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(16)$$

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(17)$$

$$\underline{a} = \left(\frac{L}{mr}\right)^2\left[\frac{\epsilon r^2}{\alpha}\sin\theta\,\frac{d}{dr}\left(\frac{\epsilon}{\alpha}\sin\theta\right) - \frac{1}{r}\right]\underline{e}_r \quad -(18)$$

$$= \left(\frac{L}{mr}\right)^2\left[\frac{\epsilon}{\alpha}\cos\theta - \frac{1}{r}\right]\underline{e}_r$$

$$= \left(\frac{L}{mr}\right)^2\left[\frac{\epsilon}{\alpha}\,\frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) - \frac{1}{r}\right]\underline{e}_r$$

i.e.

$$\boxed{\underline{a} = -\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2\underline{e}_r} \quad -(19)$$

It has been shown in previous work that the same result is obtained for eq. (9), QED. However, it is now clear that this is the result of pure kinematics, eq. (1), given eq. (16). Eq. (19) is:

$$\frac{d^2\underline{r}}{dt^2} = -\frac{1}{\alpha}\left(\frac{L}{m}\right)^2\frac{\underline{r}}{r^3} \quad -(20)$$

$$= -k\,\frac{\underline{r}}{r^3}$$

5) and this was the starting equation of note 236(3), where it was shown that eq. (20) gives eq. (16). Here it has been shown that eq. (16) gives eq. (20).

So the analysis is entirely self consistent; and also consistent with the Lagrangian analysis.

The constant k is:

$$k = MG \quad -(21)$$

The Precessing Ellipse

Here

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(22)$$

So

$$\frac{dr}{d\theta} = \frac{x\epsilon r^2}{\alpha}\sin(x\theta) \quad -(23)$$

and:

$$\underline{a} = \left(\frac{L}{mr}\right)^2\left[\frac{x\epsilon r^2}{\alpha}\sin(x\theta)\frac{d}{dr}\left(\frac{x\epsilon}{\alpha}\sin(x\theta)\right) - \frac{1}{r}\right]\underline{e}_r$$

$$= \left(\frac{L}{mr}\right)^2\left[\frac{\epsilon}{\alpha}x^2\cos(x\theta) - \frac{1}{r}\right]\underline{e}_r$$

$$= \left(\frac{L}{mr}\right)^2\left[\frac{(x^2-1)}{r} - \frac{x^2}{\alpha}\right]\underline{e}_r \quad -(24)$$

which is the same result as derived in previous work from eq. (9).

It is now clear that this is the result of the purely kinematic equation (1) given the function (22).

Eq. (24) is the equation of motion:

$$\boxed{\frac{d^2 r}{dt^2} = -\left[\frac{x^2}{\alpha} + \frac{(1-x^2)}{r}\right]\left(\frac{L}{m}\right)^2 \frac{r}{r^3}} \quad -(25)$$

This equation of motion gives the precessing conical sections and fractal conical sections.

It gives none of the errors of the Einsteinian analysis and is fully relativistic.

Conclusion

A completely general theory of all planar orbits can be derived from pure kinematics, eq. (1). All orbits are due to the spin connection $\underline{\omega}$
