230(7): New Definition of the Cartan Tetrad.

The Cartan tetrad is defined by:

$$V^a = q^a_\mu V^\mu \quad -(1)$$

where V is any vector field in any mathematical space of any dimension.

To Prove

$$q^a_\mu = \frac{V^a V_\mu}{V^\mu V_\mu} \quad -(2)$$

Proof

$$V^a = \frac{V^a V_\mu}{V^\mu V_\mu} V^\mu \quad -(3)$$

$$= V^a \frac{V_\mu V^\mu}{V^\mu V_\mu}$$

$$= V^a$$

Q.E.D.

Interpretation

By convention:

$$V^\mu V_\mu = \underline{V}^\mu \cdot \underline{V}_\mu$$

$$= \sum_{\mu=0}^{3} V^\mu V_\mu \quad -(4)$$

$$= V^0 V_0 + \dots + V^3 V_3$$

2) In eq. (2), q^a_μ is a vector valued one-form or rank two mixed index tensor, so the product $V^a V_\mu$ is also a rank two mixed index tensor. It is the general product of V^a and V_μ. On the other hand, $V^\mu V_\mu$ is a scalar, i.e the dot product of V^μ and V_μ.

Examples

Consider the unit vectors of the circular polar basis:

$$\underline{V}^{(1)} = \underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right), \quad -(5)$$

$$\underline{V}^{(2)} = \underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right), \quad -(6)$$

$$\underline{V}^{(3)} = \underline{e}^{(3)} \quad -(7)$$

The tetrad elements from these definitions are:

$$q^{(1)}_X = \frac{1}{\sqrt{2}}, \quad q^{(1)}_Y = -\frac{i}{\sqrt{2}}, \quad -(8)$$

$$q^{(2)}_X = \frac{1}{\sqrt{2}}, \quad q^{(2)}_Y = \frac{i}{\sqrt{2}} \quad -(9)$$

$$q^{(3)}_Z = 1 \quad -(10)$$

They are obtained from the definitions:

$$e^a = \left(e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)}\right) \quad -(11)$$

3) and $e^a = (e^0, e^1, e^2, e^3)$ — (12)

Here:

$$e^{(0)} = 1, \; e^{(1)} = \frac{1}{\sqrt{2}}(1-i), \; e^{(2)} = \frac{1}{\sqrt{2}}(1+i), \; e^{(3)} = 1 \quad \text{— (13)}$$

$$e^0 = 1, \; e^1 = 1, \; e^2 = 1, \; e^3 = 1. \quad \text{— (14)}$$

So:

$$\begin{bmatrix} e^{(0)} \\ e^{(1)} \\ e^{(2)} \\ e^{(3)} \end{bmatrix} = \begin{bmatrix} q^{(0)}_0 & q^{(0)}_1 & q^{(0)}_2 & q^{(0)}_3 \\ q^{(1)}_0 & q^{(1)}_1 & q^{(1)}_2 & q^{(1)}_3 \\ q^{(2)}_0 & q^{(2)}_1 & q^{(2)}_2 & q^{(2)}_3 \\ q^{(3)}_0 & q^{(3)}_1 & q^{(3)}_2 & q^{(3)}_3 \end{bmatrix} \begin{bmatrix} e^0 \\ e^1 \\ e^2 \\ e^3 \end{bmatrix}. \quad \text{— (15)}$$

So:

$$q^a_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{— (16)}$$

$$= e^a e_\mu / e^\mu e_\mu$$

It is clear that the elements of q^a_μ are scalar valued, and in this notation:

4) $$V_1^{(1)} = \frac{1}{\sqrt{2}}, \quad V_2^{(1)} = -\frac{i}{\sqrt{2}},$$
$$V_1^{(2)} = \frac{1}{\sqrt{2}}, \quad V_2^{(2)} = \frac{i}{\sqrt{2}}, \quad \text{— (17)}$$
$$V_0^{(0)} = 1, \quad V_3^{(3)} = 1.$$

Therefore, if:

$$a = (1), \mu = 1 \quad \text{— (18)}$$

then

$$\boxed{V_1^{(1)} = \frac{\underline{e}^{(1)} \cdot \underline{e}_1}{\underline{e}^1 \cdot \underline{e}_1}} \quad \text{— (19)}$$

Here:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \quad \text{— (20)}$$

$$\underline{e}_1 = - \underline{i} \quad \text{— (21)}$$

$$\underline{e}^1 = \underline{i}, \quad \text{— (22)}$$

so:

$$\underline{e}^{(1)} \cdot \underline{e}_1 = -\frac{1}{\sqrt{2}} \quad \text{— (23)}$$

$$\underline{e}^1 \cdot \underline{e}_1 = -1 \quad \text{— (24)}$$

and

$$V_1^{(1)} = \frac{1}{\sqrt{2}} \quad \text{— (25)}$$

QED.

Similarly the other scalar valued elements are:

5)

$$q^{(1)}_{2} = \frac{\underline{e}^{(1)} \cdot \underline{e}_2}{\underline{e}^{2} \cdot \underline{e}_2} = -\frac{i}{\sqrt{2}} \quad -(26)$$

$$q^{(2)}_{1} = \frac{\underline{e}^{(2)} \cdot \underline{e}_1}{\underline{e}^{1} \cdot \underline{e}_1} = \frac{1}{\sqrt{2}} \quad -(27)$$

$$q^{(2)}_{2} = \frac{\underline{e}^{(2)} \cdot \underline{e}_2}{\underline{e}^{2} \cdot \underline{e}_2} = \frac{i}{\sqrt{2}} \quad -(28)$$

$$q^{(0)}_{0} = \frac{\underline{e}^{(0)} \cdot \underline{e}_0}{\underline{e}^{0} \cdot \underline{e}_0} = 1 \quad -(29)$$

$$q^{(3)}_{3} = \frac{\underline{e}^{(3)} \cdot \underline{e}_3}{\underline{e}^{3} \cdot \underline{e}_3} = 1 \quad -(30)$$

In general, for any vector field $\underline{V}$ in any mathematical spaces labelled a and μ, the scalar valued elements of the Cartan tetrad are defined by:

$$\boxed{q^{a}_{\mu} = \frac{\underline{V}^{a} \cdot \underline{V}_{\mu}}{\underline{V}^{\mu} \cdot \underline{V}_{\mu}}} \quad -(31)$$

6)

The complete tetrad is the matrix defined by eq. (16). It is clear that the matrix should not be confused with the scalar valued elements of the matrix. The complete matrix is defined by eq. (2), the scalar valued elements of the matrix are defined by eq. (31).

Calculation of the Cartan Torsion.

This is defined by:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(32)$$

$$= \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \Omega^a_{\mu\nu}$$

where

$$\Omega^a_{\mu\nu} = \omega^a_{\mu\nu} - \omega^a_{\nu\mu} . \quad -(33)$$

So the normalized torsion is:

$$S^a_{\mu\nu} = T^a_{\mu\nu} - \Omega^a_{\mu\nu} \quad -(34)$$

$$= \partial_\mu q^a_\nu - \partial_\nu q^a_\mu$$

It is now possible to evaluate individual elements of the normalized torsion.

7) By antisymmetry:

$$\boxed{S^{a}_{\mu\nu} = 2\partial_\mu q^{a}_{\nu}} \quad -(35)$$

If $a = (1), \mu = 1, \nu = 2$ — (36)

then:

$$S^{(1)}_{12} = 2\partial_1 q^{(1)}_{2}$$

$$\boxed{S^{(1)}_{12} = 2\frac{\partial}{\partial X}\left(\frac{\underline{V}^{(1)}\cdot\underline{V}_2}{\underline{V}^{2}\cdot\underline{V}_2}\right)} \quad -(37)$$

If: $a = (1), \mu = 3, \nu = 1$ — (38)

$$S^{(1)}_{31} = 2\frac{\partial}{\partial Z}\left(\frac{\underline{V}^{(1)}\cdot\underline{V}_1}{\underline{V}^{1}\cdot\underline{V}_1}\right) \quad -(39)$$

In electrodynamics:

$$F^{(1)}_{31} = A^{(0)}\frac{\partial}{\partial Z}\left(\frac{\underline{e}^{(1)}\cdot\underline{e}_1}{\underline{e}^{1}\cdot\underline{e}_1}\right) \quad -(40)$$

where

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right)\exp\left(i(\omega t - \kappa Z)\right) \quad -(41)$$

8) The Momentum Four Vectors

In this case:

$$p^a = q^a_\mu \, p^\mu \quad -(42)$$

and the tensorial definition of the tetrad is:

$$q^a_\mu = \frac{p^a p_\mu}{p^\mu p_\mu} \quad -(43)$$

Eqs. (42) and (43) are self consistent:

$$p^a = \left(\frac{p^a p_\mu}{p^\mu p_\mu} \right) p^\mu = p^a \quad -(44)$$

Here:

$$p^\mu p_\mu = m^2 c^2 \quad -(45)$$

so

$$q^a_\mu = \frac{p^a p_\mu}{m^2 c^2} \quad -(46)$$

The momentum tetrad is:

$$p^a_\mu := p_0 \, q^a_\mu \quad -(47)$$

so

$$p^a_\mu = p_0 \left(\frac{p^a p_\mu}{m^2 c^2} \right) \quad -(48)$$

9) From eq. (42) in eq. (48):

$$p^a_\mu = \frac{p_0 \, q^a_\mu \, p^\mu p_\mu}{m^2 c^2} = p_0 \, q^a_\mu \left(\frac{p^\mu p_\mu}{m^2 c^2} \right) \quad -(49)$$

However:

$$p^\mu p_\mu = m^2 c^2 \quad -(50)$$

so eq. (49) is consistent with eq. (47).

Now multiply both sides of eq. (49) by q^μ_a, and use:

$$q^\mu_a \, q^a_\mu = 1 \quad -(51)$$

to find:

$$q^\mu_a \, p^a_\mu = \left(\frac{p_0}{m^2 c^2} \right) p^\mu p_\mu \quad -(52)$$

$$= p_0$$

so it is found that:

$$p^\mu p_\mu = m^2 c^2 \quad -(53)$$

self consistently. Finally, as in note 230(5), the vacuum produces the correction:

$$\boxed{p^\mu p_\mu = m^2 c^2 \left(1 - \frac{e A_0}{p_0} \right)} \quad -(54)$$