230(8): New Definition of the Tetrad.

This is obtained by considering the definition of the tetrad in terms of basis vectors:

$$\begin{bmatrix} \underline{e}^{(0)} \\ \underline{e}^{(1)} \\ \underline{e}^{(2)} \\ \underline{e}^{(3)} \end{bmatrix} = \begin{bmatrix} q^{(0)}_0 & q^{(0)}_1 & q^{(0)}_2 & q^{(0)}_3 \\ q^{(1)}_0 & q^{(1)}_1 & q^{(1)}_2 & q^{(1)}_3 \\ q^{(2)}_0 & q^{(2)}_1 & q^{(2)}_2 & q^{(2)}_3 \\ q^{(3)}_0 & q^{(3)}_1 & q^{(3)}_2 & q^{(3)}_3 \end{bmatrix} \begin{bmatrix} \underline{e}^0 \\ \underline{e}^1 \\ \underline{e}^2 \\ \underline{e}^3 \end{bmatrix} \quad -(1)$$

For example, consider the circular polar basis superimposed on the Cartesian basis, and for the sake of argument consider the transverse components only. Then:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(2)$$

where

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \quad -(3)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \quad -(4)$$

Multiply both sides of eq. (2) by $[\underline{i} \;\; \underline{j}]$, then:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} [\underline{i} \;\; \underline{j}] = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} [\underline{i} \;\; \underline{j}] \quad -(5)$$

$$= \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{i}\cdot\underline{i} & \underline{i}\cdot\underline{j} \\ \underline{j}\cdot\underline{i} & \underline{j}\cdot\underline{j} \end{bmatrix}$$

2)

$$= \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad - (6)$$

$$= \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} .$$

The left hand side is

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{i} & \underline{j} \end{bmatrix} = \begin{bmatrix} \underline{e}^{(1)} \cdot \underline{i} & \underline{e}^{(1)} \cdot \underline{j} \\ \underline{e}^{(2)} \cdot \underline{i} & \underline{e}^{(2)} \cdot \underline{j} \end{bmatrix} \quad - (7)$$

Therefore in general:

$$\boxed{q_\mu^a = \underline{e}^a \cdot \underline{e}_\mu^T} \quad - (8)$$

Where:

$$\underline{e}^a = \begin{bmatrix} e^{(0)} \\ \underline{e}^{(1)} \\ \underline{e}^{(2)} \\ \underline{e}^{(3)} \end{bmatrix} ; \quad \underline{e}_\mu^T = \begin{bmatrix} e^0, -\underline{e}_1, -\underline{e}_2, -\underline{e}_3 \end{bmatrix} \quad - (9)$$

Here:

$$e^{(0)} = e^0 = \underline{1} \quad - (10)$$

$$\underline{e}^{(3)} = -\underline{e}_3 = \underline{k} . \quad - (11)$$

3) The transverse tetrad components are given by:

$$q^a_\mu = \begin{bmatrix} \underline{e}^{(1)} \cdot \underline{i} & \underline{e}^{(1)} \cdot \underline{j} \\ \underline{e}^{(2)} \cdot \underline{i} & \underline{e}^{(2)} \cdot \underline{j} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \quad -(12)$$

i.e. $q^{(1)}_1 = \frac{1}{\sqrt{2}}, \quad q^{(1)}_2 = \frac{-i}{\sqrt{2}}, \quad -(13)$

$q^{(2)}_1 = \frac{i}{\sqrt{2}}, \quad q^{(2)}_2 = \frac{1}{\sqrt{2}}.$

The complete transverse tetrad vector is:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad -(14)$$

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad -(15)$$

and eqs. (13) are the components of eqns. (14) and (15).

In general, for any vectors V^a and V_μ:

$$V^a = q^a_\mu V^\mu \quad -(16)$$

and

$$\boxed{q^a_\mu = \frac{\underline{V}^a \cdot \underline{V}_\mu}{\underline{V}^\mu \cdot \underline{V}_\mu}} \quad -(17)$$

The same result as in previous notes.