230(4): Operator Definition of the Momentum Tetrad.

For the general vector field $\underline{V}$ the tetrad is defined by:

$$q^a_\mu = \frac{\underline{V}^a \cdot \underline{V}_\mu}{\underline{V}^\mu \cdot \underline{V}_\mu} . \quad -(1)$$

For the momentum field p:

$$q^a_\mu = \frac{\underline{p}^a \cdot \underline{p}_\mu}{\underline{p}^\mu \cdot \underline{p}_\mu} . \quad -(2)$$

Define the momentum tetrad as:

$$p^a_\mu = p_0 \, q^a_\mu . \quad -(3)$$

From the Einstein energy equation:

$$p^\mu p_\mu = \underline{p}^\mu \cdot \underline{p}_\mu = m^2 c^2 \quad -(4)$$

with summation implied over repeated μ indices. So:

$$\boxed{p^a_\mu = \frac{p_0}{m^2 c^2} \underline{p}^a \cdot \underline{p}_\mu} \quad -(5)$$

In quantum mechanics:

$$\boxed{\hat{p}^a_\mu \psi = \frac{p_0}{m^2 c^2} \left(\underline{\hat{p}}^a \cdot \underline{\hat{p}}_\mu \right) \psi} \quad -(6)$$

which is the Schroedinger postulate for the momentum tetrad.

In previous work the electromagnetic potential was

2) defined as:

$$A^a_\mu = A_0\, q^a_\mu \quad \text{--- (7)}$$

So it is possible to introduce the minimal prescription:

$$\boxed{p^a_\mu \rightarrow p^a_\mu + e A^a_\mu} \quad \text{--- (8)}$$

Eq (5) can be checked by using:

$$\underline{p}^a = q^a_\mu\, \underline{p}^\mu \quad \text{--- (9)}$$

so:

$$p^a_\mu = \frac{p_0}{m^2c^2}\, q^a_\mu\, \underline{p}^\mu \cdot \underline{p}_\mu \quad \text{--- (10)}$$

Multiply both sides of eq (10) by q^μ_a and use:

$$q^\mu_a\, q^a_\mu = \underline{1} \quad \text{--- (11)}$$

to find:

$$p_0 = q^\mu_a\, p^a_\mu = \left(\frac{p_0}{m^2c^2}\right) \underline{p}^\mu \cdot \underline{p}_\mu \quad \text{--- (12)}$$

$$= \frac{p_0}{m^2c^2}\, p^\nu p_\nu \quad \text{--- (13)}$$

so:

$$p^\nu p_\nu = m^2c^2, \quad \text{--- (14)}$$

<u>QED.</u> As an illustration of this theory consider (a) to define the complex circular basis and μ to

3) define the Cartesian basis. These are two different bases of the same mathematical space. The unit vectors are

$$e^{(0)} = 1, \underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}), \underline{e}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}), \underline{e}^{(3)} = \underline{k} \quad -(15)$$

$$e_0 = 1, \underline{e}_1 = -\underline{i}, \underline{e}_2 = -\underline{j}, \underline{e}_3 = -\underline{k} \quad -(16)$$

Therefore:

$$p^{(0)} = p_0, \underline{p}^{(1)} = \frac{p_0}{\sqrt{2}}(\underline{i} - i\underline{j}), \underline{p}^{(2)} = \frac{p_0}{\sqrt{2}}(\underline{i} + i\underline{j}), \underline{p}^{(3)} = p_0\underline{k} \quad -(17)$$

$$p_0 = p_0, \underline{p}_1 = -p_0\underline{i}, \underline{p}_2 = -p_0\underline{j}, \underline{p}_3 = -p_0\underline{k} \quad -(18)$$

From eq. (10):

$$\hat{p}^{(0)}_0 \phi = \left(\frac{p_0}{m^2c^2}\right) \hat{p}^{(0)} \hat{p}_0 \phi \quad -(19)$$

$$\hat{p}^{(1)}_1 \phi = \left(\frac{p_0}{m^2c^2}\right) \hat{\underline{p}}^{(1)} \cdot \underline{p}_1 \phi \quad -(20)$$

$$\hat{p}^{(2)}_2 \phi = \left(\frac{p_0}{m^2c^2}\right) \hat{\underline{p}}^{(2)} \cdot \underline{p}_2 \phi \quad -(21)$$

$$\hat{p}^{(3)}_3 \phi = \left(\frac{p_0}{m^2c^2}\right) \hat{\underline{p}}^{(3)} \cdot \underline{p}_3 \phi \quad -(22)$$

In eqs (19) to (22):

$$\hat{p}^{(0)} \hat{p}_0 = \hat{p}^0 \hat{p}_0 \quad -(23)$$

$$\hat{\underline{p}}^{(3)} \cdot \underline{p}_3 = \hat{\underline{p}}^3 \cdot \hat{\underline{p}}_3 \quad -(24)$$

4) $$\hat{\underline{p}}^{(1)} \cdot \underline{p}_1 = \frac{1}{\sqrt{2}}\left(\underline{p}^1 - i\underline{p}^2\right) \quad (25)$$

$$= \frac{1}{\sqrt{2}} \underline{p}^1 \cdot \underline{p}_1 \quad -(26)$$

$$\hat{\underline{p}}^{(2)} \cdot \underline{p}_2 = \frac{1}{\sqrt{2}}\left(\underline{p}^1 + i\underline{p}^2\right) \cdot \underline{p}_2 \quad -(27)$$

$$= \frac{i}{\sqrt{2}} \underline{p}^2 \cdot \underline{p}_2$$

Therefore:

$$\left(\hat{p}^{(0)}_0 + \sqrt{2}\,\hat{p}^{(1)}_1 - i\sqrt{2}\,\hat{p}^{(2)}_2 + \hat{p}^{(3)}_3\right)\phi$$

$$= \left(\frac{p_0}{m^2c^2}\right)\left(\hat{p}^0 \hat{p}_0 + \hat{\underline{p}}^1 \cdot \hat{\underline{p}}_1 + \hat{\underline{p}}^2 \cdot \hat{\underline{p}}_2 + \hat{\underline{p}}^3 \cdot \hat{\underline{p}}_3\right)\phi$$

$$= p_0 \phi \quad -(28)$$

So:

$$\boxed{\left(\hat{p}^{(0)}_0 + \sqrt{2}\left(\hat{p}^{(1)}_1 - i\hat{p}^{(2)}_2\right) + \hat{p}^{(3)}_3\right)\phi = p_0 \phi} \quad -(29)$$

In eq. (28), use:

$$p^\mu p_\mu = -\hbar^2 \Box = \left(\hat{p}^0 \hat{p}_0 + \hat{\underline{p}}^1 \cdot \hat{\underline{p}}_1 + \hat{\underline{p}}^2 \cdot \hat{\underline{p}}_2 + \hat{\underline{p}}^3 \cdot \hat{\underline{p}}_3\right)$$

$$-(30)$$

5) Hence:

$$-\left(\frac{\hbar^2 p_0}{m^2 c^2}\right)\Box\phi = \left(\hat{p}^{(0)}_0 + \sqrt{2}\left(\hat{p}^{(1)}_1 - i\,\hat{p}^{(2)}_2\right) + \hat{p}^{(3)}_3\right)\phi \quad \text{— (31)}$$

i.e.

$$\boxed{\Box\phi = -\left(\frac{m^2 c^2}{\hbar^2 p_0}\right)\left(\hat{p}^{(0)}_0 + \sqrt{2}\left(\hat{p}^{(1)}_1 - i\,\hat{p}^{(2)}_2\right) + \hat{p}^{(3)}_3\right)\phi} \quad \text{— (32)}$$

which is an expansion of the d'Alembertian operator in terms of linear combinations of momentum operators.
