228(7): Relativistic Generalization of Quantum Tunnelling Theory.

Consider the Einstein energy equation:

$$E^2 = \gamma^2 m^2 c^4 = p^2 c^2 + m^2 c^4, \quad -(1)$$

so:

$$\boxed{\gamma m c^2 = \frac{1}{\gamma m}\left(p^2 + m^2 c^2\right)} \quad -(2)$$

This is the necessary linearization needed for the quantum theory, i.e:

$$\hat{H}\phi = E\phi \quad -(3)$$

where

$$\hat{H} = \frac{1}{\gamma m}\left(\hat{p}^2 + m^2 c^2\right) \quad -(4)$$

$$E = \gamma m c^2 \quad -(5)$$

It follows that:

$$\boxed{\hat{p}^2 \phi = m^2 c^2 (\gamma^2 - 1)\phi} \quad -(6)$$

The basic postulate of quantum mechanics is:

$$p^\mu = i\hbar \partial^\mu = \hbar \kappa^\mu \quad -(7)$$

where:

2)

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right); \quad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right); \quad k^\mu = \left(\frac{\omega}{c}, \underline{k}\right) \quad -(8)$$

So:

$$\boxed{\hat{p}^2 \psi = \hbar^2 k^2 \psi = m^2 c^2 (\gamma^2 - 1)\psi = \left(\frac{E^2}{c^2} - m^2 c^2\right)\psi}$$

$$-(9)$$

For a free particle:

$$\boxed{k = \frac{mc}{\hbar}(\gamma^2 - 1)^{1/2}} \quad -(10)$$

In the non-relativistic limit:

$$m^2 c^2 (\gamma^2 - 1) = m^2 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)$$

$$\xrightarrow[v \ll c]{} m^2 v^2 \quad -(11)$$

so

$$p \longrightarrow mv \quad -(12)$$

For the purposes of quantum tunnelling theory denote:

$$\boxed{k = \frac{mc}{\hbar}(\gamma^2 - 1)^{1/2}} \quad -(13)$$

In the presence of a potential V_0,

3) eq. (3) becomes:

$$\hat{H}\phi = E\phi \quad -(14)$$

where $$\hat{H} = \frac{1}{\gamma m}\left(\hat{p}^2 + m^2c^2\right) + V \quad -(15)$$

so $$\hat{p}^2\phi = \left(\gamma m(E-V) - m^2c^2\right)\phi$$

$$= \hbar^2\kappa^2\phi \quad -(16)$$

and $$\boxed{\kappa^2 = \frac{1}{\hbar^2}\left(\gamma m(E-V) - m^2c^2\right)} \quad -(17)$$

Denote $$\kappa_0 = \left(\frac{mc}{\hbar}\right)^2 \quad -(18)$$

This is the Compton wavenumber. So:

$$\boxed{\kappa^2 + \kappa_0^2 = \frac{\gamma m(E-V)}{\hbar^2}} \quad -(19)$$

Eq. (13) is:

$$k^2 = \left(\frac{mc}{\hbar}\right)^2\left(\gamma^2 - 1\right) \quad -(20)$$

i.e.

4)

$$k^2 + \kappa_0^2 = \gamma^2 \left(\frac{mc}{\hbar}\right)^2 \quad -(21)$$

where

$$E = \gamma mc^2 \quad -(22)$$

so

$$\boxed{k^2 + \kappa_0^2 = \frac{\gamma m}{\hbar^2} E} \quad -(23)$$

Results

$$\boxed{\begin{aligned} k^2 + \kappa_0^2 &= \frac{\gamma m}{\hbar^2} E \\ \kappa^2 + \kappa_0^2 &= \frac{\gamma m}{\hbar^2} (E - V) \end{aligned}} \quad -(24)$$

Kinetic Energy Representation

In order to obtain a precise analogy with the Schroedinger equation, write eq. (9) as:

$$\boxed{\frac{\hat{p}^2}{2m} \phi = \frac{mc^2}{2} (\gamma^2 - 1) \phi} \quad -(25)$$

In the non relativistic limit:

$$\frac{mc^2}{2}(\gamma^2 - 1) = \frac{mc^2}{2}\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right) \quad -(26)$$

$$\xrightarrow[v \ll c]{} \frac{1}{2} mv^2$$

5)

So:

$$\frac{\hat{p}^2}{2m}\phi = \frac{1}{2}mv^2\phi. \quad -(27)$$

For a free particle:

$$E = T = \frac{1}{2}mv^2 \quad -(28)$$

so

$$\frac{\hat{p}^2}{2m}\phi = E\phi = -\frac{\hbar^2}{2m}\nabla^2\phi \quad -(29)$$

and

$$\nabla^2\phi = -\left(\frac{2mE}{\hbar^2}\right)\phi \quad -(30)$$

i.e

$$\left(\nabla^2 + k^2\right)\phi = 0 \quad -(31)$$

where

$$k^2 = \frac{2mE}{\hbar^2} \quad -(32)$$

which is the non-relativistic limit of:

$$\boxed{k^2 = \frac{2mE}{\hbar^2} \; ; \; E = \frac{mc^2}{2}\left(\gamma^2 - 1\right)} \quad -(33)$$

This is the required relativistic generalization of the k of the incoming particle in quantum tunnelling theory.

6) In the presence of a potential, eq. (25) becomes:

$$\left(\frac{\hat{p}^2}{2m} + V\right)\phi = \frac{mc^2}{2}\left(\gamma^2 - 1\right)\phi \qquad -(\cancel{27}\ 34)$$

$$= E\phi$$

So

$$-\hbar^2\nabla^2\phi = 2m\left(E - V\right)\phi \qquad -(\cancel{28}\ 35)$$

$$= \kappa^2\hbar^2\phi$$

and

$$\boxed{\kappa^2 = \frac{2m(E-V)}{\hbar^2}; \quad E = \frac{mc^2}{2}\left(\gamma^2 - 1\right)} \qquad -(\cancel{29}\ 36)$$

The relativistic transmission coefficient is:

$$= 16\kappa^2k^2 \Big/ \left(\left(e^{4\kappa a} + e^{-4\kappa a}\right)\left(k^2+\kappa^2\right) + 2\left(\kappa^4 + k^4 - 6\kappa^2k^2\right)\right) \qquad -(\cancel{30}\ 37)$$

with k and κ defined by eqs. (33) and (37).