228(5): Development of Enhanced Quantum Tunnelling Theory.

Quantum tunnelling theory is the basis of S matrix theory and quantum scattering theory. Consider the potential in Fig (1):

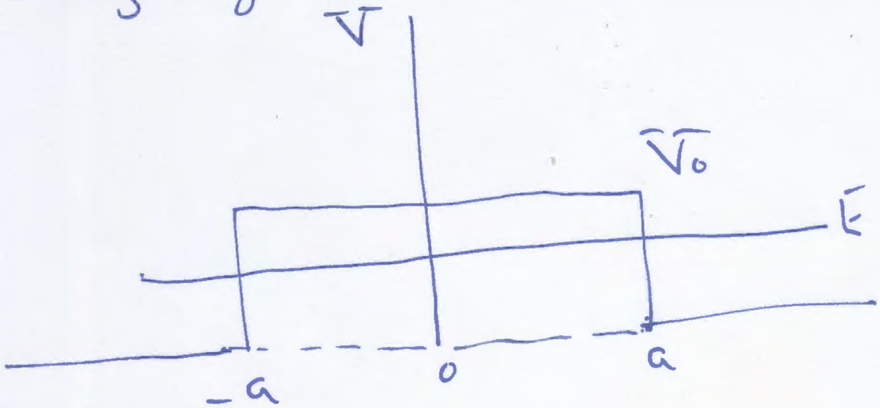

Fig (1)

The wave functions are:

$$\phi(x) = A e^{ikx} + B e^{-ikx} \quad (x < -a)$$

$$= C e^{-\kappa x} + D e^{\kappa x} \quad (-a < x < a)$$

$$= F e^{ikx} + G e^{-ikx} \quad (a < x) \qquad -(1)$$

(E. Merzbacher, "Quantum Mechanics" (Wiley, 1970, 2nd. ed.,) pp. 93 ff.)

The boundary conditions at $x = -a$ require:

$$A e^{-ika} + B e^{ika} = C e^{\kappa a} + D e^{-\kappa a} \qquad -(2)$$

$$A e^{-ika} - B e^{ika} = \frac{i\kappa}{k} \left(C e^{\kappa a} - D e^{-\kappa a} \right) \qquad -(3)$$

so:

2)

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \quad -(4)$$

where:

$$a = \frac{1}{2}\left(1 + i\frac{\kappa}{k}\right) \exp\left(\kappa a + ika\right) \quad -(5)$$

$$b = \frac{1}{2}\left(1 - i\frac{\kappa}{k}\right) \exp\left(-\kappa a + ika\right) \quad -(6)$$

$$c = \frac{1}{2}\left(1 - i\frac{\kappa}{k}\right) \exp\left(\kappa a - ika\right) \quad -(7)$$

$$d = \frac{1}{2}\left(1 + i\frac{\kappa}{k}\right) \exp\left(-\kappa a - ika\right). \quad -(8)$$

The boundary conditions at $x = a$ imply:

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \quad -(9)$$

where

$$e = \frac{1}{2}\left(1 - i\frac{k}{\kappa}\right) \exp\left(a\kappa + ika\right) \quad -(10)$$

$$f = \frac{1}{2}\left(1 + i\frac{k}{\kappa}\right) \exp\left(a\kappa - ika\right) \quad -(11)$$

$$g = \frac{1}{2}\left(1 + i\frac{k}{\kappa}\right) \exp\left(-a\kappa + ika\right) \quad -(12)$$

$$h = \frac{1}{2}\left(1 - i\frac{k}{\kappa}\right) \exp\left(-a\kappa - ika\right) \quad -(13)$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \quad -(14)$$

Eq. (14) can be checked by computer algebra. It

3) can be worked out by hand as follows:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + cg & cf + dh \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}. \quad -(15)$$

For example:

$$ae = \frac{1}{4}\left(1 + i\frac{\kappa}{k}\right)\left(1 - \frac{ik}{\kappa}\right) \exp\left(2(a\kappa + ika)\right) \quad -(16)$$

$$bg = \frac{1}{4}\left(1 - \frac{i\kappa}{k}\right)\left(1 + \frac{ik}{\kappa}\right) \exp\left(2(-\kappa a + ika)\right) \quad -(17)$$

So:

$$ae = \frac{1}{4}\left[\left(1 + \frac{i\kappa}{k}\right)\left(1 - \frac{ik}{\kappa}\right) e^{2a\kappa} + \left(1 - \frac{i\kappa}{k}\right)\left(1 + \frac{ik}{\kappa}\right) e^{-2a\kappa}\right] e^{2ika} \cdot e^{2a\kappa} \quad -(18)$$

$$= \left(\cosh(2a\kappa) + \frac{i\epsilon}{2}\sinh(2a\kappa)\right) e \quad -(19)$$

where

$$\epsilon = \frac{\kappa}{k} - \frac{k}{\kappa}. \quad -(20)$$

So

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} i & j \\ k & e \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \quad -(21)$$

$$i = \left(\cosh(2\kappa a) + i\frac{\epsilon}{2}\sinh(2\kappa a)\right) e^{2ika}$$

$$j = \frac{in}{2}\sinh(2\kappa a)$$

$$k = -\frac{in}{2}\sinh(2\kappa a)$$

4)

$$l = \left(\cosh(2\kappa a) - i\frac{\epsilon}{2}\sinh(2\kappa a)\right) e^{-2ika}$$

Consider a wave incident from the left and transmitted to the right through the barrier. There is also a reflected wave present of amplitude B. In this case:

$$G = 0 \quad -(22)$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} F \\ 0 \end{bmatrix} \quad -(23)$$

i.e

$$A = iF \quad -(24)$$

$$B = kF. \quad -(25)$$

The transmission coefficient is:

$$T = \left(\frac{F}{A}\right)^2 = \frac{e^{-4ika}}{\left(\cosh(2\kappa a) + i\frac{\epsilon}{2}\sinh(2\kappa a)\right)^2} \quad -(26)$$

If:

$$\kappa a \gg 1 \quad -(27)$$

$$T \rightarrow 16\exp(-4\kappa a)\left(\frac{k\kappa}{k^2+\kappa^2}\right)^2 \quad -(28)$$

where

$$k = \frac{(2mE)^{1/2}}{\hbar}, \quad \kappa = \frac{(2m(V_0 - E))^{1/2}}{\hbar} \quad -(29)$$

Therefore

$$k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2} \quad -(30)$$

5) and

$$\hbar \kappa = \frac{2m}{\hbar^2}\left(E(V_0 - E)\right)^{1/2}, \quad -(31)$$

so

$$\frac{k\kappa}{k^2+\kappa^2} = \frac{1}{V_0}\left(E\left(V_0 - E\right)\right)^{1/2} \quad -(32)$$

Therefore:

$$\boxed{T \to 16 \exp\left(-4\kappa a\right) \frac{E\left(V_0 - E\right)}{V_0^2} \quad -(33)}$$

for

$$\kappa a \gg 1 \quad -(34)$$

Eq. (33) is the result of the Schroedinger equation for a step barrier of type (1). In low energy nuclear reaction (LENR), we wish to maximize T for a given κ.

Using eq. (29), :

$$2m\left(V_0 - E\right) = \hbar^2 \kappa^2 \quad -(35)$$

so

$$T \to 16 \exp\left(-4\kappa a\right) \frac{E}{V_0^2} \frac{\hbar^2 \kappa^2}{2m}. \quad -(36)$$

6) Eq. (36) can be written as:

$$T = \left(\frac{8E\hbar^2}{nV_0^2 a^2}\right) x^2 e^{-4x} \quad -(37)$$

where

$$x = \kappa a. \quad -(38)$$

The maximum value of T is found when:

$$\frac{dT}{dx} = 0. \quad -(39)$$

From eq. (37):

$$\frac{dT}{dx} = \left(\frac{8E\hbar^2}{nV_0^2 a^2}\right) e^{-4x}(2x-4) \quad -(40)$$

So T is maximized when:

$$\boxed{x = \kappa a = 2}, \quad -(41)$$

i.e.

$$\boxed{\kappa = \frac{2}{a}} \quad -(42)$$

and

$$p = \hbar\kappa = \frac{2\hbar}{a} \quad -(43)$$

Therefore

$$\boxed{pa = 2\hbar} \quad -(44)$$

7)

If a is the radius of an atom then the incoming momentum of the second atom needed to maximize the transmission coefficient is given by eq. (44). If the transmission coefficient is maximized, the two atoms fuse to give a fusion product.

If the radius of the atom is 10^{-10} metres, then κ must be of the order of $10^{10}\ m^{-1}$, or $10^{8}\ cm^{-1}$.

Accurate Method

Computer algebra can be used to compute dT/dx from eq. (26):

$$T = \frac{\exp(-4ika)}{\left(\cosh(2x) + i\frac{\epsilon}{2}\sinh(2x)\right)^2} \quad -(45)$$

where from eq. (29):

$$k = \left(\frac{E}{V_0 - E}\right)^{1/2}\kappa \quad -(46)$$

so:

$$T = \frac{\exp\left(-4i\left(\frac{E}{V_0 - E}\right)^{1/2}x\right)}{\left(\cosh(2x) + i\frac{\epsilon}{2}\sinh(2x)\right)^2} \quad -(47)$$

8) where
$$\epsilon = \frac{\kappa}{k} - \frac{k}{\kappa} \quad -(48)$$

The transmission coefficient is maximized at:
$$\frac{dT}{dx} = 0. \quad -(49)$$

where
$$x = \kappa a \quad -(50)$$

This computation must be carried out in two steps, firstly the real part of T must be computed from eq. (47), then the maximum of the real part found as a function of x. This would give the κ needed to maximize the transmission coefficient T.

Finally:
$$\cancel{\hbar}\kappa_{max} = \left(2m\left(V_0 - E\right)_{max}\right)^{1/2} \quad -(51)$$

so the value of $V_0 - E$ can be found at which transmission is maximized.

In carrying out the computation,
$$\epsilon = \left(\frac{V_0 - E}{E}\right)^{1/2} - \left(\frac{E}{V_0 - E}\right)^{1/2} \quad -(52)$$

9) The total energy E of the particle tunnelling in to the barrier is constant, so k is a constant. It follows that

$$e^{-4ika} = \cos(4ka) - i\sin(4ka) \qquad -(\cancel{31})\ 53$$

is a complex constant. Therefore from eq. $(\cancel{26})$:

$$T = \frac{\left(\cos(4ka) - i\sin(4ka)\right)\left(\cosh(2\kappa a) - \frac{i\epsilon}{2}\sinh(2\kappa a)\right)^2}{\left(\cosh^2(2\kappa a) + \frac{\epsilon^2}{2}\sinh^2(2\kappa a)\right)^2} \qquad -(\cancel{32})\ 54$$

where

$$\epsilon = \frac{x}{ak} - \frac{ak}{x} \qquad -(\cancel{33})\ 55$$

So:

$$T = \frac{\left(\cos(4ka) - i\sin(4ka)\right)\left(\cosh(2x) - \frac{i\epsilon}{2}\sinh(2x)\right)^2}{\left(\cosh^2(2x) + \frac{\epsilon^2}{2}\sinh^2(2x)\right)^2} \qquad -(\cancel{34})\ 56$$

Therefore:

$$\boxed{T = \frac{\left[\cos(4ka)\left(\cosh^2(2x) - \sinh^2(2x)\right) + \epsilon\sin(4ka)\cosh(2x)\sinh(2x)\right]}{\left(\cosh^2(2x) + \frac{\epsilon^2}{2}\sinh^2(2x)\right)^2}} \qquad -(57)$$

10) where

$$\epsilon = \frac{x}{ak} - \frac{ak}{x} \quad -(58)$$

The computer algebra must work out:

$$\frac{d(\mathrm{Re}\,T)}{dx} = 0 \quad -(58)$$

and find x from this equation.

This is the conventional theory of quantum tunnelling from the Schroedinger equation, and is the base line for further development in order to find the condition for maximum transmission coefficient, then to enhance that condition. That will give the optimal condition for low energy nuclear reaction.