228(4): The Most Direct Quantization of Relativistic Classical Physics.

The usual quantization starts with the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(1)$$

which can be written as:

$$p^\mu p_\mu = m^2 c^2, \quad -(2)$$

where

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right), \quad p_\mu = \left(\frac{E}{c}, -\underline{p}\right). \quad -(3)$$

The Schroedinger postulate is:

$$p^\mu = i\hbar \partial^\mu \quad -(4)$$

where

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right). \quad -(5)$$

From eqs. (2) and (4):

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(6)$$

which is the Klein Gordon equation. The d'Alembertian is:

$$\Box := \partial^\mu \partial_\mu. \quad -(7)$$

In UFT 173 it was shown that the Klein Gordon equation can be factorized into the fermion equation, used in UFT 227.

For purposes of investigating quantum tunnelling

) in low energy nuclear reaction it is possible to use a different method of quantization, which is simpler and more direct than the usual method.

Eq. (1) is simply an algebraic re-expression of the relativistic linear momentum of a free particle:

$$\underline{p} = \gamma m \underline{v} \quad -(8)$$

where γ is the Lorentz factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(9)$$

Here m is the mass, $\underline{v}$ the velocity and c the vacuum speed of light.

Apply to eq. (8) the Schroedinger postulate:

$$\underline{p} = -i\hbar \underline{\nabla} \quad -(10)$$

so

$$-i\hbar \underline{\nabla} \phi = \gamma m \underline{v} \phi \quad -(11)$$

where ϕ is the wave function. Therefore:

$$\boxed{\underline{\nabla} \phi = i \frac{\gamma m \underline{v}}{\hbar} \phi} \quad -(12)$$

Eq. (1) may also be expressed as:

$$E = \gamma m c^2 \quad -(13)$$

Now apply the Schroedinger postulate to eq.

3) (13):

$$E = i\hbar \frac{\partial}{\partial t} \quad -(14)$$

so

$$i\hbar \frac{\partial \phi}{\partial t} = \gamma m c^2 \phi \quad -(15)$$

or

$$\boxed{\frac{\partial \phi}{\partial t} = -i \frac{\gamma m c^2}{\hbar} \phi} \quad -(16)$$

The classical relativistic Hamiltonian is:

$$H = E + V \quad -(17)$$

$$= \gamma m c^2 + V.$$

Apply the Schroedinger postulate then:

$$\hat{H} \phi = H \phi \quad -(18)$$

where

$$\hat{H} = i\hbar \frac{\partial}{\partial t} + V \quad -(19)$$

so

$$\left(i\hbar \frac{\partial}{\partial t} + V \right) \phi = H \phi \quad -(20)$$

$$= \left(\gamma m c^2 + V \right) \phi$$

Eq. (20) can be written as a first order differential equation in time:

$$i\hbar \frac{\partial \phi}{\partial t} = (H - V)\phi \quad -(21)$$

$$= E\phi$$

which is eq. (16).

Now square eq. (8):

$$p^2 = \gamma^2 m^2 v^2 \quad -(22)$$

so

$$\boxed{\nabla^2 \phi = -\left(\frac{\gamma m v}{\hbar}\right)^2 \phi} \quad -(23)$$

Square eq. (13):

$$E^2 = \gamma^2 m^2 c^4 \quad -(24)$$

so

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{\gamma m c^2}{\hbar}\right)^2 \phi} \quad -(25)$$

From eqs. (23) and (25):

$$\Box \phi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = -\left(\frac{m}{\hbar}\right)^2 \gamma^2 (c^2 - v^2)\phi$$

$$-(26)$$

In this equation:

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} = \frac{c^2}{c^2 - v^2} \quad -(27)$$

so eq. (26) is the Klein Gordon equation:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(28)$$

The Klein Gordon equation eliminates γ, but also eliminates information. It becomes clear that the use of γ is valid relativistic quantum mechanics because eqs. (26) and (28) are the same equations.

Now consider:

$$p^2 = \gamma^2 m^2 v^2, \quad -(29)$$

$$E^2 = p^2 c^2 + m^2 c^4, \quad -(30)$$

So:

$$p^2 = \frac{E^2}{c^2} - m^2 c^2 = \left(\gamma^2 - 1\right) m^2 c^2 \quad -(31)$$

The relativistic hamiltonian is:

$$H = \gamma m c^2 + V \quad -(32)$$

where V is the potential energy, so:

6)

$$\boxed{H = \frac{1}{\gamma m}\left(p^2 + m^2c^2\right) + V} \quad -(33)$$

Now apply the Schroedinger postulate:

$$p^2 = -\hbar^2 \nabla^2 \quad -(34)$$

to find the relativistic hamiltonian operator:

$$\boxed{\hat{H} = \frac{1}{\gamma m}\left(-\hbar^2\nabla^2 + m^2c^2\right) + V} \quad -(35)$$

with:

$$\hat{H}\phi = H\phi. \quad -(36)$$

Finally define:

$$* \quad \boxed{\begin{aligned}\left(\hat{H} - mc^2\right)\phi &= \left(H - mc^2\right)\phi \\ &= \left((\gamma - 1)mc^2 + V\right)\phi\end{aligned}} \quad -(37)$$

This is the correct relativistic generalization of the Schroedinger equation.

This equation seems to have been missed by Dirac, Klein and Gordon, and

7)

is the required linear generalization. It is now possible to define the energy:

$$E_1 := (\gamma - 1)mc^2 + V \quad -(38)$$

so

$$\boxed{(\hat{H} - mc^2)\phi = E_1 \phi} \quad -(39)$$

where:

$$\hat{H} - mc^2 = -\frac{\hbar^2 \nabla^2}{m}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} + V$$

$$+ mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{1/2} - 1\right) \quad -(40)$$

$$= -\frac{\hbar^2 \nabla^2}{m\gamma} + mc^2\left(\frac{1}{\gamma} - 1\right) + V$$

i.e. $$-\frac{\hbar^2}{m\gamma}\nabla^2\phi = \left(E_1 - V - mc^2\left(\frac{1}{\gamma} - 1\right)\right)\phi$$

or

$$\boxed{\nabla^2\phi = -\frac{m\gamma}{\hbar^2}\left(E_1 - V - mc^2\left(\frac{1}{\gamma} - 1\right)\right)\phi}$$

$$-(41)$$

8) In the non-relativistic limit:

$$v << c \quad -(42)$$

$$\hat{H} - mc^2 \rightarrow \frac{\hat{p}^2}{2m} + V \quad -(43)$$

and

$$H - mc^2 = (\gamma - 1)mc^2 + V$$

$$\rightarrow \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 + V \quad -(44)$$

$$\rightarrow \frac{1}{2}mv^2 + V = \frac{p^2}{2m} + V$$

in which case eq. (39) becomes the non-relativistic Schroedinger equation:

$$H\phi = E_1\phi \quad -(45)$$

Q.E.D.

Eq. (41) is the rigorously correct relativistic generalization of the Schroedinger equation:

$$\nabla^2\phi = -\frac{2m}{\hbar^2}(E - V)\phi \quad -(46)$$

used in quantum tunnelling theory.

Therefore:

9)

$$2(E - V) \rightarrow \gamma\left(E_1 - V - mc^2\left(\frac{1}{\gamma} - 1\right)\right) \quad -(47)$$

and in the ultrarelativistic limit:

$$v \rightarrow c, \quad -(48)$$

$$\gamma\left(E_1 - V - mc^2\left(\frac{1}{\gamma} - 1\right)\right) \rightarrow$$

$$\rightarrow \gamma\left(E_1 - V - mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{1/2} - 1\right)\right)$$

$$\rightarrow \gamma\left(E_1 - V + mc^2\right)$$

$$\rightarrow \infty.$$

Therefore nuclear fusion occurs in this ultra relativistic limit.