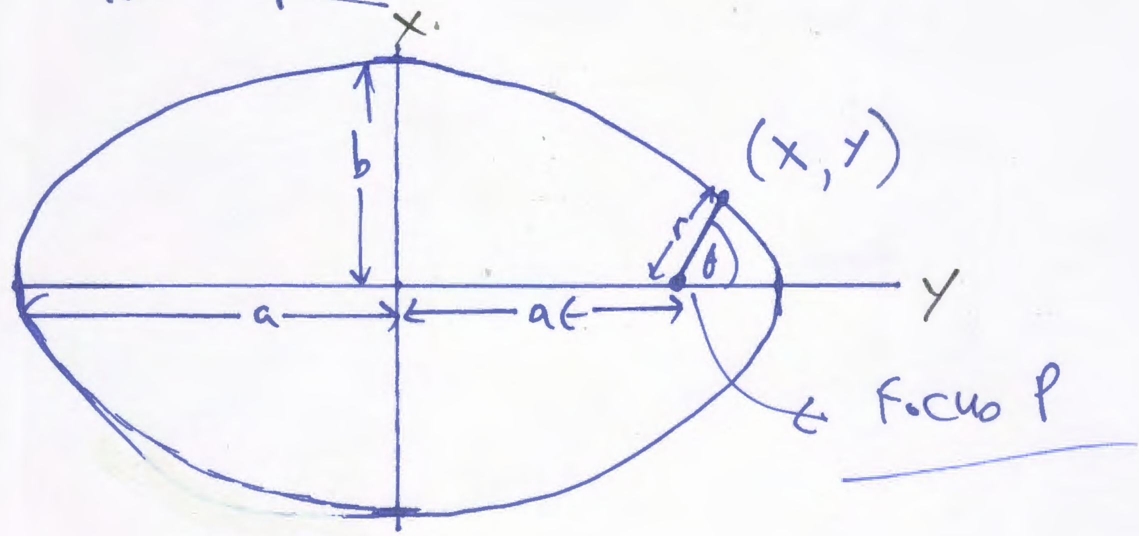
222(1): Application of Green's Theorem to the Ellipse.



The ellipse is:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad -(1)$$

i.e.

$$X = a\cos\theta, \quad Y = b\sin\theta. \quad -(2)$$

The polar equation of the ellipse with respect to the focus is obtained as follows. The focus is focus P is (aε, 0). The distance r from the point (aε, 0) to the point (X, Y) is:

$$r^2 = (X^2 - a^2\epsilon^2)^{1/2} + Y^2 \quad -(3)$$

$$= (X^2 - a^2\epsilon^2)^{1/2} + b^2\left(1 - \frac{X^2}{a^2}\right)$$

where

$$\epsilon^2 = 1 - \frac{b^2}{a^2}. \quad -(4)$$

So:

2) $$r^2 = X^2 - 2a\epsilon X + a^2\epsilon^2 + b^2 - X^2\frac{b^2}{a^2}$$

$$= X^2\left(1-\frac{b^2}{a^2}\right) - 2a\epsilon X + a^2\left(1-\frac{b^2}{a^2}\right) + b^2$$

$$= X^2\epsilon^2 - 2a\epsilon X + a^2 \quad - (5)$$

Therefore
$$r = \pm(\epsilon X - a). \quad - (6)$$

If the negative root is taken:
$$\boxed{r = -\epsilon X + a}. \quad - (7)$$

In polar coordinates:
$$\boxed{X = a\epsilon + r\cos\theta} \quad - (8)$$
$$Y = r\sin\theta \quad - (9)$$

From eq. (8) in eq. (7):
$$r = -\epsilon(a\epsilon + r\cos\theta) + a \quad - (10)$$

i.e.
$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad - (11)$$

where
$$\alpha = a(1-\epsilon^2) \quad - (12)$$

The area of the ellipse is given by an application of Green's Theorem:
$$\frac{1}{2}\oint X\,dY - Y\,dX = \int dX\,dY \quad - (13)$$

In polar coordinates:

3)

$$x = a\cos\theta, \quad y = b\sin\theta, \qquad -(14)$$

$$dx = -a\sin\theta\, d\theta, \quad dy = b\cos\theta\, d\theta \qquad -(15)$$

So:

$$A = \oint x\, dy = \int_0^{2\pi} a\cos\theta\, b\cos\theta\, d\theta$$

$$= ab \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= ab \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} \qquad -(16)$$

$$\boxed{A = \pi ab} \qquad -(17)$$

In polar coordinates, if:

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad -(18)$$

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta \qquad -(19)$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta \qquad -(20)$$

then:

$$A = \frac{1}{2}\oint (x\,dy - y\,dx)$$

$$\boxed{A = \frac{1}{2}\oint r^2\, d\theta} \qquad -(21)$$

Therefore Green's theorem is a useful way of evaluating the area. From eq. (21)

$$dA = \frac{1}{2} r^2\, d\theta \qquad -(22)$$

In astronomy, eq. (22) is Kepler's second law.

Therefore:

$$\boxed{A = \frac{1}{2}\oint r^2 d\theta = \oint X dY} \quad -(23)$$

where:

$$X = a\cos\theta, \; Y = b\sin\theta, \quad -(24)$$

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(25)$$

Check

$$A = \frac{1}{2}\oint r^2 d\theta = \frac{\alpha^2}{2}\int_0^{2\pi} \frac{d\theta}{(1+\epsilon\cos\theta)^2} \quad -(26)$$

$$= \frac{\alpha^2}{2(1-\epsilon^2)}\left[\frac{2}{(1-\epsilon^2)^{1/2}}\tan^{-1}\left(\frac{(1-\epsilon)\tan\theta/2}{(1-\epsilon^2)^{1/2}}\right) - \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\right]_0^{2\pi}$$

$$= ab\tan^{-1}\left((\tan\pi)\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}\right)$$

$$= ab\tan^{-1}0 = \pi ab, \quad \text{QED}$$

It is easier to evaluate A from:

$$A = \oint X dY, \quad -(27)$$

$$X = a\cos\theta, \; Y = b\sin\theta, \quad -(28)$$

and this is an illustration of Green's theorem.

5) For example, if A ~~needs~~ to be evaluated for a precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(29)$$

then:

$$A = \oint X\,dY \quad -(30)$$

where

$$X = a\cos(x\theta) \quad -(31)$$
$$Y = b\sin(x\theta). \quad -(32)$$

then:

$$dX = -a x\sin(x\theta)\,d\theta \quad -(33)$$
$$dY = b x\cos(x\theta)\,d\theta \quad -(34)$$

and

$$A = \oint X\,dY = \int_0^{2\pi} a b x^2\cos^2(x\theta)\,d\theta. \quad -(35)$$

Let

$$\beta = x\theta \quad -(36)$$

then

$$d\theta = \frac{1}{x}\,d\beta \quad -(37)$$

and

$$A = abx\int_0^{2\pi x}\cos^2\beta\,d\beta \quad -(38)$$

$$= abx\left(\frac{\beta}{2}+\frac{1}{4}\sin 2\beta\right)\Big|_0^{2\pi x}$$

$$\boxed{A = abx\left(\pi x + \frac{1}{4}\sin(4\pi x)\right)} \quad -(39)$$