222(4): Adjustments to the Force Law due to Precession.

The precessing conical section is described by:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos(x\theta)\right) \quad -(1)$$

Therefore:

$$\frac{dr}{dt} = -\frac{L}{\mu}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(2)$$

$$\frac{d^2 r}{dt^2} = -\left(\frac{L}{\mu}\right)^2 \frac{1}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(3).$$

From eq. (1):

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{x\epsilon}{\alpha}\sin(x\theta) \quad -(4)$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2\epsilon}{\alpha}\cos(x\theta). \quad -(5)$$

The force is defined by:

$$F(r) = \mu\frac{d^2 r}{dt^2} - \frac{L^2}{\mu r^3} \quad -(6)$$

$$= \mu\left(\frac{L}{\mu}\right)^2\frac{x^2\epsilon}{\alpha r^2}\cos(x\theta) - \frac{L^2}{\mu r^3} \quad -(7)$$

where

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(8)$$

2) Therefore:

$$F(r) = \frac{1}{\mu}\frac{L^2 x^2 \epsilon}{\alpha r^2}\frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) - \frac{L^2}{\mu r^3} \quad -(9)$$

$$= \frac{L^2}{\mu r^3}\left(x^2 - 1\right) - \frac{L^2}{\mu r^2}\frac{x^2}{\alpha}$$

i.e. $$F(r) = \frac{L^2}{\mu r^2}\left(\frac{x^2 - 1}{r} - \frac{x^2}{\alpha}\right) \quad -(10)$$

If $$x = 1 \quad -(11)$$

this reduces to eq. (22) of note 222(3):

$$F(r) = -\frac{L^2}{\mu \alpha r^2} = -\frac{mMG}{r^2} \quad -(12)$$

QED.

Graphical Analysis

1) Plot:
$$\frac{d^2 r}{dt^2} = \left(\frac{xL}{\mu}\right)\frac{1}{r^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right) \quad -(13)$$

and:

2)
$$\frac{dr}{dt} = \left(\frac{x\epsilon L}{\alpha\mu}\right)\sin(x\theta) \quad -(14)$$

) where:

$$\sin(x\theta) = \left(1 - \cos^2(x\theta)\right)^{1/2} \quad -(15)$$

$$= \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2}$$

$$= \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha - r}{r}\right)^2\right)^{1/2}$$

$$= \left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2} / (\epsilon r)$$

So:

$$\frac{dr}{dt} = \left(\frac{x\epsilon L}{\alpha\mu}\right)\frac{\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2}}{\epsilon r} \quad -(16)$$

$$\frac{dr}{dt} = \frac{xL}{\alpha\mu r}\left(\epsilon^2 r - (\alpha - r)^2\right)^{1/2} \quad -(17)$$

$$\frac{d^2 r}{dt^2} = \left(\frac{xL}{\mu}\right)\frac{1}{r^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right)$$

Here:

$$\alpha^2 = ab\left(1 - \epsilon^2\right)^{3/2}, \quad -(18)$$

$$r_{min} = a(1 - \epsilon) = \frac{\alpha}{1 + \epsilon} \quad -(19)$$

$$r_{max} = a(1 + \epsilon) = \frac{\alpha}{1 - \epsilon} \quad -(20)$$