222(3): Meaning of the Newtonian Force Law

The Lagrangian of relevance is:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \quad -(1)$$

and the two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(2)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} . \quad -(3)$$

Using these equations:

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U}{\partial r} = F(r) \quad -(4)$$

$$L = \mu r^2 \dot{\theta} . \quad -(5)$$

Now use the substitution:

$$u = 1/r \quad -(6)$$

to find:

$$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} , \quad -(7)$$

where

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta} . \quad -(8)$$

$$= \frac{\mu r^2}{L}\frac{dr}{dt} .$$

So

$$\boxed{\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\mu}{L}\frac{dr}{dt}} \quad -(9)$$

Therefore:

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{L}\frac{dr}{dt}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{L}\frac{dr}{dt}\right)$$

$$= -\frac{\mu}{L}\frac{dt}{d\theta}\frac{d^2r}{dt^2} \quad -(10)$$

$$\boxed{\frac{d^2u}{d\theta^2} = -\frac{\mu^2}{L^2}r^2\frac{d^2r}{dt^2}} \quad -(11)$$

Therefore:

$$\frac{d^2r}{dt^2} = -\frac{L^2}{\mu^2}\frac{1}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(12)$$

and from eq. (5):

$$r\frac{d\theta}{dt} = \frac{L^2}{\mu^2}\left(\frac{1}{r}\right)^3. \quad -(13)$$

It follows that:

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{L^2}r^2F(r)} \quad -(14)$$

where

$$F(r) = -\frac{\partial U(r)}{\partial r} \quad -(15)$$

is the <u>force</u>.

In Newtonian dynamics:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad -(16)$$

So:

$$\frac{dr}{dt} = \left(\frac{L\epsilon}{\mu\alpha}\right)\sin\theta, \quad -(17)$$

$$\frac{d^2r}{dt^2} = \left(\frac{L}{\mu}\right)^2\frac{\epsilon}{\alpha}\frac{1}{r^2}\cos\theta. \quad -(18)$$

Here:

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(19)$$

so

$$\boxed{\frac{d^2r}{dt^2} = \frac{L^2}{\mu^2 r^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right)} \quad -(20)$$

Eqs. (11) and (14) give:

$$\boxed{F(r) = \mu\frac{d^2r}{dt^2} - \frac{L^2}{\mu r^3}} \quad -(21)$$

Finally eqs. (20) and (21) give:

$$F(r) = -\frac{L^2}{\mu\alpha r^2} \quad -(22)$$

where

$$L^2 = \alpha\mu k \quad -(23)$$

4)

So:

$$F(r) = -\frac{k}{r^2} = -\frac{mMG}{r^2} \quad -(24)$$

$$U(r) = -\frac{mMG}{r} \quad -(25)$$

This is the rigorously correct representation of Newtonian dynamics.

It is seen that the force in eq. (21) contains two terms. If there were no angular motion then there would be no interaction between m and M and then:

$$F(r) = m\frac{d^2 r}{dt^2} \quad -(26)$$

The lagrangian dynamics are more general than the Newtonian dynamics because the former correctly describe rotational motion. The familiar eq. (24) is the result of cancellation of terms in eq. (21). This was first shown in an earlier paper using purely geometrical arguments.

In order to repeat this analysis for a precessing conical section:

5)

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos(x\theta)\right). \quad -(27)$$

So:

$$\frac{dr}{dt} = -\frac{L}{\mu}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(28)$$

$$\boxed{\frac{dr}{dt} = -\frac{xL\epsilon}{\mu\alpha}\sin(x\theta)} \quad -(29)$$

Similarly:

$$\frac{d^2r}{dt^2} = -\frac{L^2}{\mu^2}\frac{1}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(30)$$

$$\boxed{\frac{d^2r}{dt^2} = x^2\left(\frac{L}{\mu}\right)^2\frac{\epsilon}{\alpha}\frac{1}{r^2}\cos(x\theta)} \quad -(31)$$

Graphical Analysis

Plot $\frac{dr}{dt}$ and $\frac{d^2r}{dt^2}$ against θ

in a polar plot. Here:

$$\frac{1}{r^2} = \frac{1}{\alpha^2}\left(1 + \epsilon\cos(x\theta)\right)^2 \quad -(32)$$

The results will characterize all precessing conical sections, and dr/dt is observable.

6) In general for any curve:

$$dA = \frac{1}{2} r^2 d\theta. \quad -(33)$$

In Newtonian dynamics:

$$dA = \frac{L}{2\mu} dt \quad -(34)$$

(Kepler's second law)

Therefore:

$$\frac{dA}{dr} = \frac{dA}{dt}\frac{dt}{dr} = \frac{\alpha}{2\epsilon \sin\theta} \quad -(35)$$

Here

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(36)$$

and

$$\sin^2\theta + \cos^2\theta = 1 \quad -(37)$$

so

$$\sin\theta = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(38)$$

$$= \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha - r}{r}\right)^2\right)^{1/2}$$

$$= \left(\frac{\epsilon^2 r^2 - (\alpha - r)^2}{\epsilon^2 r^2}\right)^{1/2}$$

$$\sin\theta = \left(\frac{\epsilon^2 r^2 - (\alpha - r)^2}{\epsilon^2 r^2}\right)^{1/2}$$

Therefore:

$$dA = \frac{\alpha}{2\epsilon} \frac{\epsilon r}{(\epsilon^2 r^2 - (\alpha - r)^2)^{1/2}} dr \quad - (39)$$

$$= \frac{\alpha r}{2(\epsilon^2 r^2 - (\alpha - r)^2)^{1/2}} dr$$

Therefore:

$$dr = \frac{2(\epsilon^2 r^2 - (\alpha - r)^2)^{1/2}}{\alpha r} dA \quad - (40)$$

So

$$A = \int dA = \frac{\alpha}{2} \int \frac{r}{(\epsilon^2 r^2 - (\alpha - r)^2)^{1/2}} dr \quad - (41)$$

The area of the ellipse is:

$$A = \pi ab = \pi \alpha^2 (1 - \epsilon^2)^{-3/2} \quad - (42).$$

If R is the circumference of the ellipse then:

$$\frac{\alpha}{2} \int_0^R \frac{r}{(\epsilon^2 r^2 - (\alpha - r)^2)^{1/2}} dr = \pi ab$$

$$= \pi \alpha^2 (1 - \epsilon^2)^{-3/2} \quad - (43)$$

So

$$\int_0^R \frac{r}{(\epsilon^2 r^2 - (\alpha - r)^2)^{1/2}} dr = \frac{2\pi\alpha}{(1 - \epsilon^2)^{3/2}} \quad - (44)$$
