222(2): Worked Example of the x Theory of All Orbits.

Consider the logarithmic spiral orbit:

$$r = r_0 e^{a\theta} \quad -(1)$$

In the x theory this is represented as a precessing conical section:

$$r = r_0 e^{a\theta} = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(2)$$

The force law responsible for eq. (1) is found from the Lagrangian theory and the equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L^2} F(r) \quad -(3)$$

where

$$L = \mu r^2 \frac{d\theta}{dt} \quad -(4)$$

is the conserved total angular momentum and where μ is the reduced mass. From eqs (1) and (3):

$$F(r) = -\frac{L^2}{\mu r^3}\left(1 + a^2\right) \quad -(5)$$

So a log spiral orbit is given by an inverse cube force law.

From eq. (4):

2)

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \quad -(6)$$

so:

$$t = \left(\frac{\mu r_0^2}{2aL}\right) e^{2a\theta} \quad -(7)$$

In general:

$$dA = \frac{1}{2} r^2 \, d\theta \quad -(8)$$

where A is the area, so:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{L}{2\mu} \quad -(9)$$

$$= \text{constant}$$

Therefore

$$dt = \frac{2\mu}{L} dA = \frac{\mu}{L} r^2 d\theta \quad -(10)$$

i.e

$$\boxed{t = \frac{\mu}{L} \int r^2 d\theta} \quad -(11)$$

This is a very useful equation that is true for all orbits in a plane.

From eqs. (1) and (11): $\quad -(12)$

$$t = \frac{\mu r_0^2}{L} \int e^{2a\theta} d\theta = \left(\frac{\mu r_0^2}{2aL}\right) e^{2a\theta}$$

which is eq. (7), QED.

3) In this case the angle θ is easily found in terms of t:

$$e^{2a\theta} = \left(\frac{2aL}{\mu r_0^2}\right) t \quad -(13)$$

$$\theta = \frac{1}{2a} \log_e \left(\left(\frac{2aL}{\mu r_0^2} \right) t \right) \quad -(14)$$

and this angle is what is observed in astronomy

Finally:

$$r = \left(\frac{2aLt}{\mu r_0^2} \right)^{1/2} \quad -(15)$$

From eq. (2):

$$1 + \epsilon \cos\theta = \frac{\alpha}{r_0} e^{-a\theta} \quad -(16)$$

So

$$\cos x\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r_0} e^{-a\theta} - 1 \right) \quad -(17)$$

$$= \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1 \right)$$

So:

$$x = \frac{1}{\theta} \cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1 \right) \right) \quad -(18)$$

4)

where: $$\theta = \frac{1}{a} \log_e \frac{r}{r_0} \quad -(19)$$

Finally:

$$\boxed{x = \frac{a}{\log_e\left(\frac{r}{r_0}\right)} \cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right)\right)} \quad -(20)$$

Therefore a log spiral orbit can be represented by a precessing conical section given eq. (20). In this case x must be a function of r. Eq. (20) can be graphed by computer and its properties studied graphically.