217(7) : Definition of Ellipticity for Ellipse and Hyperbola.

For the hyperbola:

$$\epsilon = \left(1 + \frac{b^2}{a^2}\right)^{1/2} \quad -(1)$$

For the ellipse:

$$\epsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2} \quad -(2)$$

where:

$$a = \frac{k}{2|E|} \quad -(3)$$

$$b = \frac{L}{(2m|E|)^{1/2}} \quad -(4)$$

So for the ellipse:

$$\epsilon = \left(1 - \frac{2EL^2}{mk^2}\right)^{1/2}, \quad -(5)$$

$$\epsilon < 1. \quad -(6)$$

For the hyperbola:

$$\epsilon = \left(1 + \frac{2EL^2}{mk^2}\right)^{1/2} \quad -(7)$$

$$\epsilon > 1$$

Note that Marion and Thornton use the second the formula for the hyperbola in

2) part of their eq. (7.40)

The basic formula is:

$$\theta(r) = \int \frac{L}{r^2}\left(2m\left(E + \frac{k}{r} - \frac{L^2}{2mr^2}\right)\right)^{-1/2} dr \quad -(8)$$

I will check whether Marion and Thornton have evaluated this correctly.

The parabola is:

$$\frac{\alpha}{r} = 1 + \cos\theta \quad -(9)$$

i.e.

$$\epsilon = 1 \quad -(10)$$

In this case

$$E = 0 \quad -(11)$$

On page 258, Marion and Thornton make the classification:

Conic Section	Ellipticity	Energy
Hyperbola	$\epsilon > 1$	$E > 0$
Parabola	$\epsilon = 1$	$E = 0$
Ellipse	$0 < \epsilon < 1$	$V_{min} < E < 0$
Circle	$\epsilon = 0$	$E = V_{min}$

3) The negative value of the total energy E arise only because:

$$V(r) := 0 \text{ at } r = \infty \quad -(8)$$

where

$$V(r) = -\frac{k}{r} + \frac{L^2}{2mr^2} \quad -(9)$$

$V(r)$ is a fictitious function in Newtonian dynamics because it combines the real potential $V(r)$ with energy coming from the angular motion about the centre of force.

Taking E as negative is equivalent to using eq (5) for the ellipse.