217(11): Comparison of Potentials.

In gravitational theory:

$$u(r) = -\frac{kx^2}{r} + \frac{(x^2-1)k\alpha}{2r^2} \quad -(1)$$

where:

$$k = mMG, \quad \alpha = \frac{L^2}{mk}. \quad -(2)$$

For the H atom in the Schroedinger equation:

$$u(r) = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{l(l+1)\hbar^2}{2mr^2} \quad -(3)$$

From eqs. (1) and (3), assume that:

$$kx^2 = \frac{e^2}{4\pi\epsilon_0} \quad -(4)$$

$$(x^2-1)k\alpha = l(l+1)\hbar^2/m. \quad -(5)$$

Here:

$$e = 1.60219 \times 10^{-19} \; C$$

$$4\pi\epsilon_0 = 1.11265 \times 10^{-10} \; J^{-1} C^2 m^{-1}$$

m = electron mass = 9.10953×10^{-31} kg

M = proton mass = 1.67265×10^{-27} kg

$$G = 6.67265 \times 10^{-11} \; Nm^2 kg^{-2}$$

$$= 6.67265 \times 10^{-11} \; m^3 s^{-2} kg^{-1}$$

So

$$x^2 = \frac{e^2}{4\pi\epsilon_0 m M G} = \frac{C^2}{C^2 J^{-1} m^{-1} kg^2 m^3 s^{-2} kg^{-1}}$$

$$= J\,kg^{-1} m^{-2} s^{-2}$$

$$= \text{unitless} \quad -(6)$$

2)

So:

$$\boxed{x = 4.474467 \times 10^{19}} \quad - (7)$$

for the H atom.

From eqs. (2) and (5):

$$L^2 (x^2 - 1) = \ell(\ell+1)\hbar^2 \quad - (8)$$

so to an excellent approximation:

$$L^2 = \ell(\ell+1)\left(\frac{\hbar}{x}\right)^2 \quad - (9)$$

$$L = \sqrt{\ell(\ell+1)}\,\frac{\hbar}{x} \quad - (10)$$

The magnitude of the angular momentum in quantum mechanics is:

$$L_0 = \sqrt{\ell(\ell+1)}\,\hbar \quad - (11)$$

so

$$\boxed{L = \frac{L_0}{x}} \quad - (12)$$

Under the conditions (7) and (12) the two potentials (1) and (2) are the same.

The gravitational Schroedinger equation may then be deduced as follows.

3) The original Schroedinger equation for H is:

$$\hat{H}\phi = E\phi \quad -(13)$$

where $$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \quad -(14)$$

From eq. (4):

$$\frac{e^2}{4\pi\epsilon_0} \rightarrow mMGx^2. \quad -(15)$$

From eq (5):

$$\frac{\hbar^2}{2m} \rightarrow \frac{(x^2-1)L^2}{2m\,l(l+1)} = \left(\frac{x^2-1}{x^2}\right)\frac{\hbar^2}{2m} \quad -(16)$$

using eq. (12)

So the gravitational Schroedinger equation is:

$$\hat{H}\phi = E\phi \quad -(17)$$

where: $$\hat{H} = -\left(\frac{x^2-1}{x^2}\right)\frac{\hbar^2}{2m}\nabla^2 - x^2\frac{mMG}{r} \quad -(18)$$

Conversely the Schroedinger equation becomes the orbital equation:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(19)$$

under the conditions (4), (5) and (11)