215(7): Classical Description of the Precession of the Perihelion.

Lagrangian dynamics gives the force equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L}F(r) \quad -(1)$$

where the conserved total angular momentum is:

$$L = mr^2\dot{\theta}. \quad -(2)$$

The orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(3)$$

where

$$2\alpha = \text{latus rectum}$$
$$\epsilon = \text{ellipticity}$$
$$x = \text{precession constant}$$

From eq. (3):

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha}r^2\sin(x\theta) \quad -(4)$$

so:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{x\epsilon L}{\alpha m}\sin(x\theta) \quad -(5)$$

The orbital velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(6)$$

so

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2. \quad -(7)$$

It follows that:

$$v^2 = \left(\frac{x L \epsilon}{\alpha m}\right)^2 \sin^2(x\theta) + \left(\frac{L}{mr}\right)^2 \quad -(8)$$

This is already a valid classical description of the precession of the perihelion. This equation can be tested experimentally. The observables are:

v = linear velocity of the planet
ω = $d\theta / dt$ = angular velocity of the planet
m = mass of the planet
2α = latus rectum of the orbit
ϵ = ellipticity of the orbit
x = precession constant
r = distance from sun to planet.

Testing eq. (8) should be straightforward with contemporary astronomy.

Eq. (8) results in a small correction to Kepler's equation for v as follows.

Note that:

$$1 + \epsilon \cos(x\theta) = \frac{\alpha}{r} \quad -(9)$$

so

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(10)$$

Now use:

$$\sin^2(x\theta) + \cos^2(x\theta) = 1 \quad -(11)$$

So:

$$\sin^2(x\theta) = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(12)$$

Therefore:

$$v^2 = \left(\frac{L}{m}\right)^2\left[\left(\frac{x\epsilon}{\alpha}\right)^2\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) + \frac{1}{r^2}\right] \quad -(13)$$

$$= \left(\frac{L}{m}\right)^2\left[\frac{x^2\epsilon^2}{\alpha^2} - \frac{x^2}{\alpha^2}\left(\frac{\alpha}{r} - 1\right)^2 + \frac{1}{r^2}\right]$$

$$= \left(\frac{L}{m}\right)^2\left[\frac{x^2\epsilon^2}{\alpha^2} - \frac{x^2}{\alpha^2}\left(\frac{\alpha - r}{r}\right)^2 + \frac{1}{r^2}\right]$$

$$= \left(\frac{L}{m}\right)^2\left[\frac{x^2\epsilon^2}{\alpha^2} + \frac{1}{r^2}\left(1 - \frac{x^2}{\alpha^2}(\alpha - r)^2\right)\right]$$

$$= \left(\frac{L}{m}\right)^2\left[\frac{x^2\epsilon^2}{\alpha^2} + \frac{1}{r^2}\left(\frac{\alpha^2 - x^2(\alpha - r)^2}{\alpha^2}\right)\right]$$

$$= \left(\frac{L}{m}\right)^2\left[\frac{x^2\epsilon^2}{\alpha^2} + \frac{1}{r^2}\left(\frac{\alpha^2 - x^2(\alpha^2 - 2\alpha r + r^2)}{\alpha^2}\right)\right]$$

$$= \left(\frac{L}{m}\right)^2\left[\frac{x^2\epsilon^2}{\alpha^2} + \frac{1}{r^2}(1 - x^2) + \frac{2x^2}{\alpha r} - \left(\frac{x}{\alpha}\right)^2\right]$$

$$= \left(\frac{L}{m\alpha}\right)^2\left[x^2(\epsilon^2 - 1) + \frac{2x^2\alpha}{r} + \frac{\alpha^2}{r^2}(1 - x^2)\right]$$

Therefore:

$$* \quad \boxed{v^2 = \left(\frac{L}{m\alpha}\right)^2\left[\frac{2x^2\alpha}{r} - x^2(1 - \epsilon^2) + \frac{\alpha^2}{r^2}(1 - x^2)\right]} \quad -(14)$$

4)

The Newtonian result is given by:

$$x = 1 \qquad -(15)$$

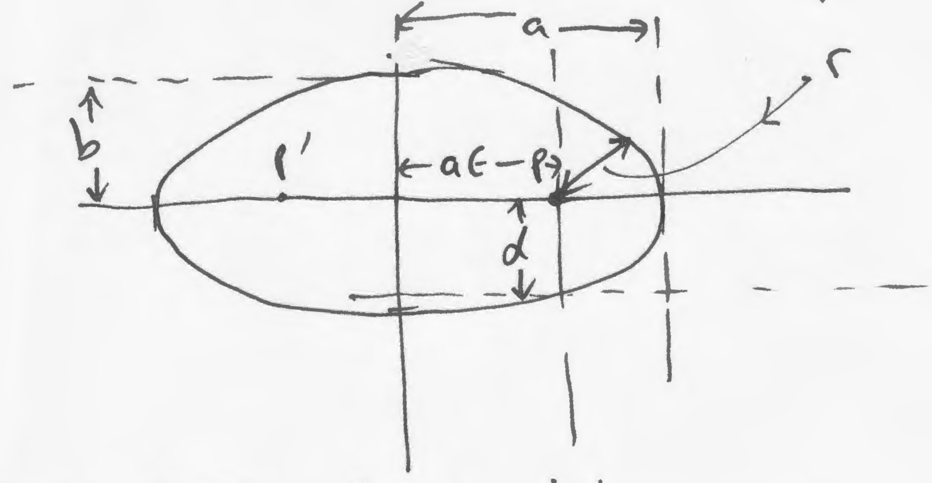
so

$$v_N^2 = \left(\frac{L}{m\alpha}\right)^2 \left[\frac{2\alpha}{r} - \left(1 - \epsilon^2\right)\right] \qquad -(16)$$

which is eq. (7) of note 215(6) QED.

Comparing eqs. (14) and (16) it is seen that the precession of the perihelion produces an additional term in $1/r^2$ and modifies the other two terms by x^2.

The ellipse is defined as follows.



The semi major axis is:

$$a = \frac{\alpha}{1 - \epsilon^2} \qquad -(17)$$

and the semi minor axis is:

$$b^2 = \frac{\alpha^2}{1 - \epsilon^2} \qquad -(18)$$

5)

As the ellipse precesses according to eq. (3) it does not change its shape, so eqs. (17) and (18) are also true for a precessing ellipse.

So eq. (14) is:

$$v^2 = \left(\frac{L}{m\alpha}\right)^2\left[\frac{2x^2\alpha}{r}\right] - \frac{x^2}{b^2}\left(\frac{L}{m}\right)^2 + \left(\frac{L}{m}\right)^2\left(\frac{1-x^2}{r^2}\right)$$

$$v^2 = \left(\frac{2x^2L^2}{m^2\alpha}\right)\frac{1}{r} + \left(\frac{L}{m}\right)^2\left(\frac{1-x^2}{r^2} - \frac{x^2}{b^2}\right)$$

i.e

$$v^2 = \left(\frac{L}{m}\right)^2\left[\frac{2x^2}{\alpha}\cdot\frac{1}{r} + \frac{1-x^2}{r^2} - \frac{x^2}{b^2}\right] \quad -(19)$$

Finally α can be eliminated using:

$$\alpha = a(1-\epsilon^2) \quad -(20)$$

so

$$\boxed{v^2 = \left(\frac{L}{m}\right)^2\left[\frac{2x^2}{(1-\epsilon^2)a}\cdot\frac{1}{r} + \frac{1-x^2}{r^2} - \frac{x^2}{b^2}\right]} \quad -(21)$$

which is the best suited for astronomy.

To test this by astronomy, a and b

6) are measured using distances of closest approach:

$$r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \qquad -(22)$$

and furthest away:

$$r_{max} = a(1+\epsilon) = \frac{\alpha}{1-\epsilon} \qquad -(23)$$

These are simultaneous equations for a and ϵ or α and ϵ. What are actually observed are r_{min} and r_{max}. The precession constant x can be found by the precession of the perihelion. Finally the quantity L/m has to be found. It is given by:

$$\frac{L}{m} = r^2\omega \qquad -(24)$$

The distance r and the angular velocity ω of the planet have to be measured. The linear velocity v of the planet has to be measured.

The precession of the perihelion of a planet in the solar system is due to Lagrangian dynamics, not to general relativity.