215(5): Lagrangian Theory of Planetary Precession

Consider the Lagrangian:

$$\mathcal{L} = T - U \qquad -(1)$$

where T is the kinetic energy and U the potential energy. In cylindrical polar coordinates in a plane:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 \right) \qquad -(2)$$

Assume that the potential energy is central, i.e. depends only on r:

$$U = U(r) \qquad -(3)$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \qquad -(4)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \qquad -(5)$$

The angular momentum is a constant of motion and is defined by:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \qquad -(6)$$

So

$$\frac{dL}{dt} = 0. \qquad -(7)$$

From eqs (1) and (5):

$$L = m r^2 \dot{\theta} = m r^2 \frac{d\theta}{dt} \qquad -(8)$$

2) and
$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(9)$$

The Hamiltonian is defined as the total energy E:
$$H = E = T + U \quad -(10)$$

Using eqs. (4) and (5) it may be shown that:
$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L} F(r) \quad -(11)$$

where the force is:
$$F(r) = -\frac{\partial U(r)}{\partial r} \quad -(12)$$

Eq. (11) is a general result of lagrangian dynamics valid for any central potential.

Note carefully that lagrangian dynamics are more general than Newtonian dynamics, the latter being defined by:
$$U(r) = -\frac{k}{r} \quad -(13)$$

where
$$k = mMG. \quad -(14)$$

Here the mass m is in orbit around the mass M, and G is Newton's constant. Eq. (11) is true for any planar orbit.

Consider the processing elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad - (15)$$

where

2α = latus rectum

ϵ = ellipticity

x = precession constant.

From eqs. (11) and (15):

$$F(r) = -\frac{k}{r^2}\left(x^2 + (1 - x^2)\frac{\alpha}{r}\right) \quad - (16)$$

which is <u>the</u> <u>correct</u> <u>result</u> of <u>Lagrangian</u> <u>dynamics</u>.

This <u>alone</u> is enough to show that Einsteinian general relativity is incorrect, because EGR gives a sum of inverse square and inverse fourth power terms.

From eq. (16):

$$\boxed{U(r) = -\frac{kx^2}{r} - \frac{k(1 - x^2)\alpha}{2r^2}} \quad - (17)$$

This is the <u>correct</u> potential, giving eq. (15).

EGR is <u>incorrect</u>, it gives a sum of inverse r and inverse cube terms. This sum does <u>not</u> give the orbit (15).

4) From eqs. (1) and (17):

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2+\dot{\theta}^2 r^2\right) + \frac{x^2 k}{r} + \frac{(1-x^2)\alpha k}{2r^2} \quad -(18)$$

$$= \frac{1}{2}m\,\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{x^2 k}{r} + \frac{(1-x^2)\alpha k}{2r^2}.$$

This can so written as:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{L_1^2}{2mr^2} + \frac{k_1}{r} \quad -(19)$$

where

$$k_1 = x^2 k = x^2 m M G \quad -(20)$$

and

$$L_1^2 = L^2 - mk\alpha(1-x^2) \quad -(21)$$

Now define eq. (19) as:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2+\dot{\beta}^2 r^2\right) - U(r) \quad -(22)$$

where:

$$L_1 = mr^2\frac{d\beta}{dt} \quad -(23)$$

$$U(r) = -\frac{k_1}{r} \quad -(24)$$

and

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\beta}} \quad -(25)$$

$$L_1 = \frac{\partial\mathcal{L}}{\partial\dot{\beta}} \quad -(26)$$

The Hamiltonian equivalent to eq. (22) is:

5)
$$H = E = \frac{1}{2} m \left(\dot{r}^2 + \dot{\beta}^2 r^2 \right) + U(r) \quad -(27)$$

So:
$$E = \frac{1}{2} m \dot{r}^2 + \frac{L_1^2}{mr^2} - \frac{k_1}{r} \quad -(28)$$

Therefore:
$$\dot{r} = \frac{dr}{dt} = \left(\frac{2}{m} \left(E + \frac{k_1}{r} - \frac{L_1^2}{mr^2} \right) \right)^{1/2} \quad -(29)$$

Now use:
$$\frac{d\beta}{dr} = \frac{dt}{dr} \frac{d\beta}{dt} = \frac{L_1}{mr^2} \frac{dt}{dr} \quad -(30)$$

From eqs. (29) and (30):
$$\beta(t) = \int \frac{L_1}{r^2} \left(2m \left(E + \frac{k_1}{r} - \frac{L_1^2}{2mr^2} \right) \right)^{-1/2} dr \quad -(31)$$

Integrating gives:
$$\cos\beta = \frac{\left(\frac{L_1^2}{mk_1} \right) \frac{1}{r} - 1}{\left(1 + \frac{2EL_1^2}{mk_1^2} \right)^{1/2}} \quad -(32)$$

Eq. (15) and (32) are ~~Ch~~ same precessing ellipse.

6)

Here:

$$\alpha = \frac{L_1^2}{m k_1}, \quad \epsilon = \left(1 + \frac{2EL_1^2}{m k_1^2}\right)^{1/2} \quad -(33)$$

$$\boxed{\begin{array}{l}\underline{\text{Conclusion}} \\ \quad \text{It has been shown that Lagrangian} \\ \text{dynamics gives a precessing elliptical orbit.}\end{array}}$$

This theory is preferred to EGR by Okham's Razor of philosophy. It also shows that EGR is plainly wrong.

The kinematics of the orbit are described by:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(34)$$

$$\underline{a} = \dot{\underline{v}} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(35)$$

For Newtonian dynamics

$$r = \frac{\alpha_0}{1 + \epsilon_0 \cos\theta} \quad -(36)$$

where

$$\alpha_0 = \frac{L^2}{mk}, \quad \epsilon_0 = \left(1 + \frac{2EL^2}{mk^2}\right)^{1/2} \quad -(37)$$

7) In this case:

$$\dot{r} = \frac{L\epsilon}{md}\sin\theta\,, \quad \dot{\theta} = \frac{L}{mr^2}. \quad -(38)$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \quad -(39)$$

For Newtonian dynamics:

$$\underline{a} = -\frac{L^2}{dm^2r^2}\,\underline{e}_r \quad -(40)$$

$$= -\frac{MG}{r^2}\,\underline{e}_r \quad -(41)$$

and

$$\underline{F} = -\frac{mMG}{r^2}\,\underline{e}_r. \quad -(42)$$

The force is directed towards M along $\underline{e}_r$.

However:

$$\underline{v} = \frac{L\epsilon}{md}\sin\theta\,\underline{e}_r + \frac{L}{mr}\,\underline{e}_\theta \quad -(43)$$

and the total orbital linear velocity $\underline{v}$ has an angular component as well as a radial component.

The problem with Newtonian dynamics is that it is defined only for motion in a straight line. When it is applied to

8) rotational motion it fails. This can be seen from the fact that the force in eq. (42) is not counterbalanced, so m would fall into M. Eq. (43) is the result of Lagrangian dynamics. The dynamics of Lagrange and Hamilton are more general than those of Newton.

The usual fallacy of textbook orbital theory is to take the rotational part of the kinetic energy:

$$T_{rot} = \frac{1}{2} m \dot{\theta}^2 r^2 = \frac{L^2}{2mr^2} \quad -(38)$$

and to define it as the effective centrifugal potential energy:

$$U_c =? \frac{L^2}{2mr^2}. \quad -(39)$$

The "effective potential" is then:

$$V = -\frac{k}{r} + \frac{L^2}{2mr^2}. \quad -(40)$$

This is just a contrivance which ultimately fails, because eq. (42) ~~means~~ that the orbit collapses.