215(3): Self Consistent Calculation of Angular Momentum

Fixed Frame

$$\underline{L} = \underline{r} \times \underline{p} = \text{constant} \quad -(1)$$

where

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(2)$$

$$\underline{p} = m\underline{v} = m\left(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta\right) -(3)$$

so

$$\underline{L} = mr^2\dot{\theta}\,\underline{e}_r \times \underline{e}_\theta = mr^2\dot{\theta}\,\underline{k} \quad -(4)$$

i.e.

$$\boxed{\underline{L} = mr^2\omega\,\underline{k}} \quad -(5)$$

which is the same result as the Lagrangian method.

Moving Frame

$$\underline{L}_m = \underline{r} \times \underline{p}_m \quad -(6)$$

where

$$\underset{\sim}{r} = r\,\underline{e}_{rm} \quad -(7)$$

$$\underline{p}_m = m\underline{v}_m = m\left(\dot{r}\,\underline{e}_{rm} + r\dot{\beta}\,\underline{e}_\beta\right)$$

$$-(8)$$

i.e.

$$\underline{L}_m = mr^2\dot{\beta}\,\underline{e}_{rm} \times \underline{e}_\beta \quad -(9)$$

$$= x m r^2\dot{\theta}\,\underline{k}$$

$$\boxed{\underline{L}_m = x m r^2\dot{\theta}\,\underline{k}} \quad -(10)$$

which is the same result as the Lagrangian method.

2) Both $\underline{L}$ and $\underline{L}_m$ are constants of motion, so

$$\boxed{\underline{L} - \underline{L}_m = (1-x)\, m r^2 \omega \underline{k} = \text{constant}.} \quad -(11)$$

So the torque is both forms is zero.

This is checked by the fact that for a central force such as:

$$\underline{F} = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(12)$$

then

$$\underline{r} = r\,\underline{e}_r \quad -(13)$$

and

$$\boxed{\underline{Tq} = \underline{F} \times \underline{r} = \underline{0}} \quad -(14)$$

Eq. (11) is the simplest explanation of the precession of a planet, and eq. (14) means that orbits are defined by force between the planet and the sun, and not by a torque. This is of course the classical theory, where space and time are still considered as separate concepts.