215(6): Calculation of the Orbital Linear Velocity for the Elliptical Orbit:

The linear velocity is given by:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta . \quad -(1)$$

From previous notes:

$$\dot{r} = \left(\frac{L\epsilon}{m\alpha}\right)\sin\theta, \quad \dot{\theta} = \frac{L}{mr^2} \quad -(2)$$

for the orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} . \quad -(3)$$

and the lagrangian:

$$\mathcal{L} = \frac{1}{2}mv^2 - U(r) . \quad -(4)$$

So:

$$v^2 = \left(\frac{L\epsilon}{m\alpha}\right)^2 \sin^2\theta + \left(\frac{L}{mr}\right)^2 \quad -(5)$$

where

$$\sin^2\theta = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 . \quad -(6)$$

So:

$$v^2 = \left(\frac{L}{m}\right)^2 \left[\left(\frac{\epsilon}{\alpha}\right)^2 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) + \frac{1}{r^2}\right]$$

$$= \left(\frac{L}{m}\right)^2 \left[\frac{\epsilon^2}{\alpha^2} + \frac{1}{r^2} - \frac{1}{\alpha^2}\left(\frac{\alpha}{r} - 1\right)^2\right]$$

$$v^2 = \left(\frac{L}{m\alpha}\right)^2 \left(2\frac{\alpha}{r} - \left(1 - \epsilon^2\right)\right) \quad -(7)$$

2) Now use:

$$\alpha = \frac{L^2}{mk}, \quad a = \frac{\alpha}{1-\epsilon^2} \quad -(8)$$

to find:

$$\boxed{v^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)} \quad -(9)$$

This is eq. (7.72) of Marion and Thornton, third edition. They obtain it from Kepler's third law and Kepler's equation. Here a is the semi major axis of the ellipse:

$$\frac{(x + a\epsilon)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(10)$$

This means that our method gives the correct result (9) from eqs. (1) and (3). The Lagrangian method we have used is much more elegant than the Kepler method of two hundred years earlier, although Kepler's method is a tour de force for its time.