215(9): General Version of Note 215(4).

The calculations of note 215(4) are summarized as follows:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(1)$$

$$L = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt}. \quad -(2)$$

$$= \frac{x\epsilon}{\alpha} r^2 \sin(x\theta) \quad -(3)$$

therefore
$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta)$$

and
$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta)\frac{L}{mr^2} \quad -(4)$$

$$\dot{r} = \frac{xL\epsilon}{m\alpha}\sin(x\theta) \quad -(5)$$

$$\dot{\theta} = \frac{L}{mr^2}. \quad -(6)$$

The second derivatives are evaluated as follows. Firstly:

$$\ddot{r} = \frac{xL\epsilon}{m\alpha}\frac{d}{dt}\left(\sin(x\theta)\right) \quad -(7)$$

$$= \frac{xL\epsilon}{m\alpha}\frac{d\sin(x\theta)}{d\theta}\frac{d\theta}{dt}$$

$$\ddot{r} = \frac{x^2L^2\epsilon}{m^2\alpha r^2}\cos(x\theta) \quad -(8)$$

Secondly:
$$\ddot{\theta} = \frac{d}{dt}\left(\frac{L}{mr^2}\right) = \frac{L}{m}\frac{d}{dt}\left(\frac{1}{r^2}\right)$$

$$= \frac{L}{m}\frac{d}{dr}\left(\frac{1}{r^2}\right)\frac{dr}{dt} \quad -(9)$$

So:

$$\ddot{\theta} = -\frac{2L}{mr^3}\cdot\frac{xL\epsilon}{md}\sin(x\theta)$$

$$\ddot{\theta} = -\frac{2L^2 x\epsilon}{m^2 r^3 d}\sin(x\theta) \qquad -(10)$$

The acceleration is:

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \qquad -(11)$$

Here:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{2L^2\epsilon x}{m^2 d r^2}\sin(x\theta) + \frac{2L^2\epsilon x}{m^2 d r^2}\sin(x\theta) \qquad -(12)$$

$$= 0$$

and

$$\ddot{r} - r\dot{\theta}^2 = \frac{x^2 L^2\epsilon}{m^2 d r^2}\cos(x\theta) - \frac{L^2}{m^2 r^3} \qquad -(13)$$

From eq. (1):

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{d}{r} - 1\right) \qquad -(14)$$

so:

$$\ddot{r} - r\dot{\theta}^2 = \frac{x^2 L^2}{m^2 d r^2}\left(\frac{d}{r} - 1\right) - \frac{L^2}{m^2 r^3} \qquad -(15)$$

$$= \left(x^2 - 1\right)\frac{L^2}{m^2 r^3} - \frac{x^2 L^2}{m^2 d r^2}$$

3)

The acceleration is therefore:

$$\underline{a} = \left(\frac{L}{m}\right)^2 \left(\frac{(x^2-1)}{r^3} - \frac{x^2}{\alpha r^2} \right) \underline{e}_r \quad -(16)$$

Finally the force is:

$$\boxed{\underline{F} = m\,\underline{a} = \frac{L^2}{m} \left(\frac{(x^2-1)}{r^3} - \frac{x^2}{\alpha r^2} \right) \underline{e}_r} \quad -(17)$$

The orbit (1) is given by the force (17).

If

$$x = 1 \quad -(18)$$

then

$$\underline{F} = -\frac{L^2}{m\alpha} \cdot \frac{1}{r^2}\, \underline{e}_r . \quad -(19)$$

In this case there is no precession and the force is an attractive inverse square law. The well known inverse square law is given by:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(20)$$

so:

$$\underline{F} = -\frac{mMG}{r^2}\, \underline{e}_r \quad -(21)$$

Note that eq. (20) can be used only if $x = 1$.

4) Eq. (17) is obtained elegantly from the Lagrangian equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \quad -(22)$$

as follows. Firstly:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos(x\theta)\right) \quad -(23)$$

so

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{x\epsilon}{\alpha}\sin(x\theta), \quad -(24)$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2\epsilon}{\alpha}\cos(x\theta)$$

Therefore

$$F(r) = -\frac{L^2}{mr^2\alpha}\left(1 + (1-x^2)\epsilon\cos(x\theta)\right) \quad -(25)$$

$$= -\frac{L^2}{mr^2\alpha}\left(1 + (1-x^2)\left(\frac{\alpha}{r} - 1\right)\right)$$

$$= -\frac{x^2L^2}{mr^2\alpha} + \frac{(x^2-1)L^2}{mr^3}$$

i. e.

$$\boxed{\underline{F} = \frac{L^2}{m}\left(\frac{(x^2-1)}{r^3} - \frac{x^2}{\alpha r^2}\right)\underline{e}_r} \quad -(26)$$

which is the same as eq. (17). Eq. (22) shows automatically that F is radial