215(8): Expression for Polar Angle θ in Terms of r for a Precessing Elliptical orbit.

The orbit is described by:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(1)$$

so

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta)$$

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(2)$$

Therefore:

$$\frac{d\theta}{dr} = \frac{\alpha}{x\epsilon}\frac{1}{r^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2} \quad -(3)$$

and

$$\boxed{\theta = \frac{\alpha}{x\epsilon}\int \frac{1}{r^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2} dr} \quad -(4)$$

The Newtonian result is given by:

$$x = 1 \quad -(5)$$

For any curve:

$$dA = \frac{1}{2} r^2 d\theta \quad -(6)$$

where dA is the infinitesimal of area. From eqs. (3) and (6):

2) From eq. (4):

$$\beta = x\theta = \frac{\alpha}{\epsilon}\int \frac{1}{r^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2\right)^{-1/2} dr \quad -(7)$$

and from eq. (1)

$$\beta = \cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right)\right) \quad -(8)$$

From the lagrangian theory of note 215(5):

$$\alpha = \frac{L_1^2}{m k_1} \; ; \; \epsilon = \left(1+\frac{2EL_1^2}{m k_1^2}\right)^{1/2} \quad -(9)$$

where

$$L_1 = x m r^2 \frac{d\theta}{dt} \quad -(10)$$

$$k_1 = x^2 m M G \quad -(11)$$

The precession of the perihelion is given by:

$$\boxed{x = \frac{1}{\theta}\cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right)\right)} \quad -(12)$$

and deflection of light is given by:

$$\boxed{\theta = \frac{1}{x}\cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right)\right)} \quad -(13)$$

<u>Both cases described by lagrangian theory.</u>

3) If r is initially R_0 and finally ∞ then:

$$\boxed{\Delta\theta = \frac{1}{x}\left(\cos^{-1}\left(-\frac{1}{\epsilon}\right) - \cos^{-1}\left(\frac{\alpha}{R_0} - 1\right)\right) \quad -(14)}$$

with α and ϵ described by eq. (9).

The photon mass m can be found from this Lagrangian method.