212(7): Equivalence of Active and Passive Rotations

In general the Christoffel connection is defined by:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad -(1)$$

The equivalence of active and passive rotations was defined in UFT199 by:

$$\underbrace{\partial_\mu V^\nu}_{\text{active}} = \underbrace{\Gamma^\nu_{\mu\lambda} V^\lambda}_{\text{passive}} \quad -(2)$$

For a rotation about the Z axis in the XY plane:

$$\boxed{\underbrace{\frac{\partial X'}{\partial Y}}_{\text{active}} = \underbrace{\Gamma^1_{23} Z'}_{\text{passive}}} \quad -(3)$$

Here:

$$\frac{\partial X'}{\partial Y} = \sin\theta \quad -(4)$$

$$\Gamma^1_{23} = \frac{\epsilon^1_{23} \sin\theta}{Z'} \quad -(5)$$

and

$$\boxed{\Gamma^1_{23} = -\Gamma^1_{32}} \quad -(6)$$

It is obvious that the connection is antisymmetric.

2) From eq. (3):

$$\left(\frac{d}{d\theta} \frac{\partial X'}{\partial Y} \right)_{\theta = 0} = 1 = \epsilon^1{}_2 = \epsilon_{12} \quad -(7)$$

where

$$\epsilon_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad -(8)$$

and

$$\left(\frac{d}{d\theta} \Gamma^1{}_{23} Z' \right)_{\theta = 0} = \epsilon^1{}_{23} \epsilon_3 = \epsilon^1{}_{23} \epsilon^3 \quad -(9)$$

where

$$\epsilon^1{}_{23} = 1 \quad -(10)$$

$$\epsilon^3 = 1 \quad -(11)$$

So eq. (3) is:

$$\boxed{\epsilon^i{}_j = \epsilon^i{}_{jk} \epsilon^k} \quad -(12)$$

This is the well known equation defining the antisymmetric unit tensor in terms of the Levi-Civita tensor and axial unit vector. The metric is:

$$g_{ij} = g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(13)$$

It follows that:

$$\epsilon^i{}_j = g^{ik} \epsilon_{kj} = \epsilon_{ij} \quad -(14)$$

$$\epsilon^i = g^{ik}\epsilon_k = \epsilon_i \quad -(15)$$

and

$$\epsilon^i{}_{jk} = g^{il}\epsilon_{ljk} = \epsilon_{ijk} \quad -(16)$$

and:

$$\epsilon_{ij} = \epsilon_{ijk}\epsilon_k \quad -(17)$$

Here:

$$\epsilon_{123} = 1 = -\epsilon_{132} = -\epsilon_{213} = \epsilon_{231}$$
$$= -\epsilon_{321} = \epsilon_{312} \quad -(18)$$

Eqs. (12) or (17) are well known, and are usually interpreted as the duality of an axial unit vector with its antisymmetric unit tensor. However, they can also be interpreted as the equivalence of an active and passive rotation.
