212(5): Illustrations of Coordinate Transformation with Rotation and Lorentz Boost.

Consider a rotation about the Z axis in a plane:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad -(1)$$

then

$$X' = X\cos\theta + Y\sin\theta \quad -(2)$$

$$Y' = -X\sin\theta + Y\cos\theta. \quad -(3)$$

Therefore:

$$\frac{\partial X'}{\partial X} = \cos\theta, \quad \frac{\partial X'}{\partial Y} = \sin\theta \quad -(4)$$

$$\frac{\partial Y'}{\partial X} = -\sin\theta, \quad \frac{\partial Y'}{\partial Y} = \cos\theta. \quad -(5)$$

Also:

$$\frac{\partial X}{\partial Y} = \tan\theta. \quad -(6)$$

Using the chain rule:

$$\frac{\partial X'}{\partial Y} = \frac{\partial X'}{\partial X}\frac{\partial X}{\partial Y} \quad -(7)$$

because:

$$X' = X'(X(Y)). \quad -(8)$$

Eq. (7) is correct because:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}. \quad -(9)$$

In this case the Christoffel connection is zero

2) because the space is Euclidean. Therefore:

$$\Gamma^{\lambda}_{\mu\nu} = 0 . \quad -(10)$$

This result must be true in all frames of reference, so:

$$\Gamma^{\lambda'}_{\mu'\nu'} = 0 \quad -(11)$$

It follows that:

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) = 0 \quad -(12)$$

It is known that:

$$\frac{\partial X}{\partial X'} \neq 0, \quad \frac{\partial Y'}{\partial Y} \neq 0, \quad \frac{\partial Y'}{\partial X} \neq 0 \quad -(13).$$

so for eq. (12) to be true:

$$\lambda = 3 \quad -(14)$$

because:

$$\frac{\partial Z}{\partial Z'} = 0, \quad \frac{\partial Y'}{\partial Z} = 0 \quad -(15)$$

$$x^1 = X, \quad x^2 = Y, \quad x^3 = Z. \quad -(16)$$

In order to understand this result recall that the active rotation of eq. (1) is equivalent to the passive rotation (UFT199):

3)

$$\underline{i}' = \underline{i}\cos\theta - \underline{j}\sin\theta \quad -(17)$$

$$\underline{j}' = \underline{i}\sin\theta + \underline{j}\cos\theta \quad -(18)$$

i.e. $$\begin{bmatrix} \underline{i}' \\ \underline{j}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(19)$$

Denote $$R_z = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad -(20)$$

then:

$$\epsilon_{ij} = \left(\frac{dR_z}{d\theta}\right)_{\theta=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad -(21)$$

The rotation generator is:

$$J_z = -i\,\epsilon_{ij} \quad -(22)$$

the antisymmetric unit tensor ϵ_{ijk} is defined by:

$$\epsilon_{ijk} = \epsilon_{ij}\,\epsilon_k \quad -(23)$$

and $$\epsilon_{ij} = \epsilon_{ijk}\,\epsilon_k \quad -(24)$$

In three dimensions this result may be lowered:

$$\epsilon_{ij} = \epsilon^k{}_{ij}\,\epsilon_k \quad -(25)$$

Therefore $$\boxed{\Gamma^k{}_{ij} = \epsilon^k{}_{ij}} \quad -(26)$$

4)

For a Z axis rotation this is exactly the type of connection defined by eqs. (12) to (16)

In three dimensions the totally anti-symmetric unit vector is an antisymmetric connection, describing a passive rotation equivalent to an active rotation.

We have:

$$\boxed{\Gamma^{k}_{ij} = - \Gamma^{k}_{ji}} \quad -(27)$$

and the connection is <u>antisymmetric</u> <u>Q.E.D.</u>

The inhomogeneous term (12) in the coordinate transform of Γ^{k}_{ij} <u>vanishes</u> because:

$$\boxed{\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = \frac{\partial x^{2'}}{\partial x^{3}} = 0} \quad -(28)$$

<u>Q.E.D.</u>

It is well known that:

$$[K_x, K_y] = -i J_z \quad -(29)$$

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where K_x and K_y are Lorentz boost generators, so:

$$\epsilon_{ij} = [K_i, K_j] \quad -(30)$$

<u>and the commutator of Lorentz boosts is also defined by the connection (27).</u>