212(6): Rotation Generators for Active and Passive Rotations

For an active rotation the generator is:

$$(J_Z)_a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad -(1)$$

and for a passive rotation it is:

$$(J_Z)_p = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad -(2)$$

So

$$(J_Z)_a = (J_Z)_p^{-1} \quad -(3)$$

because

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(4)$$

The antisymmetric unit tensor for a passive rotation is

$$\epsilon_{ij} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\epsilon_{ji} \quad -(5)$$

and is defined by:

$$\epsilon_{ij} = \epsilon^{k}_{ij}\, \epsilon_{k} \quad -(6)$$

$$= \epsilon_{ijk}\, \epsilon_{k}$$

In order to define covert units:

$$\Gamma^{k}_{ij} = \frac{1}{r}\, \epsilon^{k}_{ij} \quad -(7)$$

), Then the units of the connection are inverse metres.

Here:

$$r = \left(X^2 + Y^2\right)^{1/2} \quad -(8)$$

$$= \text{constant}$$

The unit tensor ϵ_{ij} is defined by

$$\left(\epsilon_{ij}\right)_p = \left(\frac{dR_{zp}}{d\theta}\right)_{\theta=0} \quad -(9)$$

where

$$R_{zp} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad -(10)$$

and

$$\begin{bmatrix} \underline{i}' \\ \underline{j}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(11)$$

Therefore:

$$\left(\epsilon_{12}\right)_p = -\left(\frac{d(\sin\theta)}{d\theta}\right)_{\theta=0} = -1 \quad -(12)$$

$$\left(\epsilon_{21}\right)_p = \left(\frac{d(\sin\theta)}{d\theta}\right)_{\theta=0} = 1 \quad -(13)$$

and

$$\left(\epsilon_{12}\right)_p = -\left(\epsilon_{12}\right)_a \quad -(14)$$

$$\left(\epsilon_{21}\right)_p = -\left(\epsilon_{21}\right)_a \quad -(15)$$

For an active rotation:

$$\cos\theta = \frac{\partial X'}{\partial X} = \frac{\partial Y'}{\partial Y} \quad -(16)$$

3)

So:

$$(\epsilon_{12})_p = -(\cos\theta)_{\theta=0} = -1$$

$$= -\left(\frac{\partial X'}{\partial X}\right)_{\theta=0} = -\left(\frac{\partial Y'}{\partial Y}\right)_{\theta=0} \quad -(17)$$

and $$(\epsilon_{12})_a = \left(\frac{\partial X'}{\partial X}\right)_{\theta=0} = \left(\frac{\partial Y'}{\partial Y}\right)_{\theta=0} \quad -(18)$$

In eq. (6):

$$\boxed{\left(\frac{\partial X'}{\partial X}\right)_{\theta=0} = \epsilon^{3}_{12}\,\epsilon_3} \quad -(19)$$

or:

$$\left(\frac{\partial X'}{\partial X}\right)_{\theta=0} = \epsilon^{Z}_{XY}\,\epsilon_Z \quad -(20)$$

This equation is a special case of:

$$\partial_\mu V^\nu = \Gamma^\nu_{\mu\lambda} V^\lambda \quad -(21)$$

which is the theorem used in UFT199

The transformation law for eq. (7) is:

$$\Gamma^{k'}_{i'j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\Gamma^k_{ij} - \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial}{\partial x^i}\left(\frac{\partial x^{k'}}{\partial x^j}\right)$$

$$-(22)$$

t)

The inhomogeneous term is zero because:

$$\Gamma^{3'}_{1'2'} = \frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^{3'}}{\partial x^3}\Gamma^3_{12} - \frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial}{\partial x^1}\left(\frac{\partial x^{3'}}{\partial x^2}\right) \quad -(23)$$

and

$$\boxed{\frac{\partial x^{3'}}{\partial x^2} = \frac{\partial Z'}{\partial Y} = 0} \quad -(24)$$

The transformed connection is non-zero if and only if:

$$x^{3'} = x^3 \quad -(25)$$

because rotation is being considered in the plane XY, so:

$$Z' = Z. \quad -(26)$$

There is no functional dependence of Z on X or Y, so eq. (24) follows.

Conclusions

1) The connection is antisymmetric.
2) The inhomogeneous term is zero.

The general rotation needs three rotation generators, J_X, J_Y, J_Z as is well known. In each case conclusions (1) and (2) are true