212(3): Illustration of Coordinate Transformation with the Lorentz Transform.

Consider the definition of the covariant derivative, a definition inferred by Christoffel in the eighteen sixties:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad -(1)$$

where $\Gamma^\nu_{\mu\lambda}$ is the Christoffel connection. Note that $D_\mu V^\nu$ is defined to transform as a tensor. The partial four derivative does not transform as a tensor:

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) \quad -(2)$$

and the Leibnitz Theorem gives: $-(3)$

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right) V^\nu .$$

The second term on the RHS means that the format of the partial four derivative is not maintained, so it does not transform as a tensor. The connection is introduced so that there exists a covariant derivative, a derivative such that $D_\mu V^\nu$ transforms covariantly. In general the connection transforms

2)
as:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) \quad -(4)$$

So:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial x^{\nu'}}{\partial x^{\nu}} \right) V^{\nu}$$

$$+ \left( \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) \right) V^{\lambda'}$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left( \partial_{\mu} V^{\nu} + \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \Gamma^{\nu}_{\mu\lambda} V^{\lambda'} \right) \quad -(5)$$

$$+ \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left( \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial x^{\nu'}}{\partial x^{\nu}} \right) V^{\nu} - \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) V^{\lambda'} \right)$$

The required tensor transformation law is:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} D_{\mu} V^{\nu} \quad -(6)$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left( \partial_{\mu} V^{\mu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \right)$$

Note that eq. (5) is:

3) Eqs. (6) and (7) are the same if:

$$\Gamma^{\nu}_{\mu\lambda} V^{\lambda} = \Gamma^{\nu}_{\mu\lambda} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} V^{\lambda'} \quad -(8)$$

i.e.

$$V^{\lambda} = \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} V^{\lambda'} \quad -(9)$$

<u>which is true</u>, and if:

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) V^{\nu} = \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}}\right) V^{\lambda'} \quad -(10)$$

To prove eq. (10) the indices of summation are changed:

$$\lambda \rightarrow \nu, \quad -(11)$$

so:

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) V^{\nu} = \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) \frac{\partial x^{\nu}}{\partial x^{\nu'}} V^{\nu'} \quad -(12)$$

<u>which is true</u> because:

$$V^{\nu} = \frac{\partial x^{\nu}}{\partial x^{\nu'}} V^{\nu'}. \quad -(13)$$

It is now known that:

$$\Gamma^{\nu}_{\mu\lambda} = -\Gamma^{\nu}_{\lambda\mu} \quad -(14)$$

$$\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = 0 \quad -(15)$$

because:

$$\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = \frac{\partial x^{\nu'}}{\partial x^{a}}\frac{\partial x^{a}}{\partial x^{\lambda}}$$

$$= \frac{\partial x^{a}}{\partial x^{b}}\frac{\partial x^{b}}{\partial x^{\lambda}} \quad -(16)$$

$$= 0$$

because

$$\frac{\partial x^{a}}{\partial x^{b}} = q^{a}_{\ \mu}\, q^{\mu}_{\ b} = 0 \quad -(17)$$

On the other hand:

$$\frac{\partial x^{\nu'}}{\partial x^{\nu}} \neq 0 \quad -(18)$$

because of eq. (3).

So from eq. (10):

$$\boxed{\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) = 0} \quad -(19)$$

is the only possible solution. It is shown in the next note that this is obeyed automatically by the Lorentz transform.

5) It is only possible to arrive at these results using the tangent spacetime of Cartan, and by introducing the constraints:

$$V^a = q^a_\mu V^\mu \quad -(20)$$

and

$$q^a_\mu q^\mu_b = \delta^a_b . \quad -(21)$$

So from eq. (18):

$$x^a = q^a_\mu x^\mu \quad -(22)$$

and

$$\frac{\partial x^a}{\partial x^\mu} = q^a_\mu \quad -(23)$$

so:

$$\frac{\partial x^a}{\partial x^b} = \frac{\partial x^a}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^b} = q^a_\mu q^\mu_b . \quad -(24)$$

These constraints imply eqs. (14), (15) and (19).

Hitherto these results were unknown, and they imply that:

$$\boxed{\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu V^\nu} \quad -(25)$$