212(4): Considerations of Coordinate Transform of the Connection

As in previous notes the connection transforms as:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) \quad -(1)$$

This transform is carried out in the base manifold without consideration of the tangent spacetime introduced by Cartan. Other aspects of Riemann geometry show that:

$$\Gamma^{\nu}_{\mu\lambda} = - \Gamma^{\nu}_{\lambda\mu} \quad -(2)$$

in any frame of reference, so:

$$\Gamma^{\nu'}_{\mu'\lambda'} = - \Gamma^{\nu'}_{\lambda'\mu'}. \quad -(3)$$

Partial derivatives commute, so:

$$\frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) = \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x^{\nu'}}{\partial x^{\mu}} \right) \quad -(4)$$

so the only possible solution is:

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) = 0. \quad -(5)$$

In eq. (5):

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \neq 0, \quad \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \neq 0. \quad -(6)$$

so

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}}\right)=0. \quad -(7)$$

The Cartan tangent spacetime is introduced through:

$$\Gamma^{\nu}_{\mu\lambda}=\Gamma^{a}_{\mu\lambda}\,q^{\nu}_{a} \quad -(8)$$

$$\Gamma^{a}_{\mu\lambda}=q^{a}_{\nu}\,\Gamma^{\nu}_{\mu\lambda} \quad -(9)$$

in which the tetrad is defined by:

$$x^{a}=q^{a}_{\nu}\,x^{\nu}. \quad -(10)$$

Therefore x^a is a function of x^ν. The transformation equation (1) then becomes:

$$\Gamma^{a'}_{\mu'\lambda'}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{a'}}{\partial x^{a}}\Gamma^{a}_{\mu\lambda}-\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{a'}}{\partial x^{\lambda}}\right) \quad -(11)$$

In this expression:

$$\frac{\partial x^{a'}}{\partial x^{\lambda}}=\frac{\partial x^{a'}}{\partial x^{\nu'}}\frac{\partial x^{\nu'}}{\partial x^{\lambda}}=q^{a'}_{\nu'}\frac{\partial x^{\nu'}}{\partial x^{\lambda}}$$

$$=\frac{\partial x^{a'}}{\partial x^{b'}}\frac{\partial x^{b'}}{\partial x^{\nu'}}\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \quad -(12)$$

By definition:

$$\frac{\partial x^{a'}}{\partial x^{b'}} = \frac{\partial x^{a'}}{\partial x^{\nu'}} \frac{\partial x^{\nu'}}{\partial x^{b'}} \quad \text{— (13)}$$

$$= q^{a'}_{\nu'} \, q^{\nu'}_{b'}$$

$$= \delta^{a'}_{b'}$$

Therefore if $a' \neq b'$ — (14)

$$\frac{\partial x^{a'}}{\partial x^{\lambda}} = 0 \quad \text{— (15)}$$

and from eq. (10):

$$\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = 0. \quad \text{— (16)}$$

Q.E.D

Notes

In using the chain rule:

$$\frac{\partial x^{a'}}{\partial x^{\nu'}} = \frac{\partial x^{a'}}{\partial x^{b'}} \frac{\partial x^{b'}}{\partial x^{\nu'}}, \quad \text{— (17)}$$

it has been assumed that:

$$x^{a'} = x^{a'}\left(x^{b'}\left(x^{\nu'}\right)\right) \quad \text{— (18)}$$

i.e. $x^{a'}$ is a function of $x^{b'}$ which is a function of $x^{\nu'}$. By definition:

$$x^{a'} = q^{a'}_{\ b'} x^{b'} \quad \text{— (19)}$$

So $x^{a'}$ is a function of $x^{b'}$.

This chain rule cannot be used for $\partial x^{\mu}/\partial x^{\mu'}$ and $\partial x^{\lambda}/\partial x^{\lambda'}$ because they are not functions of ~~x~~ x^{a}. The
So $\partial x^{\mu}/\partial x^{\mu'}$ and $\partial x^{\lambda}/\partial x^{\lambda'}$ are not zero.
right chain rule for $\partial x^{a'}/\partial x^{a}$ is:

$$\frac{\partial x^{a'}}{\partial x^{a}} = \frac{\partial x^{a'}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{a}} \quad \text{— (20)}$$

i.e.
$$q^{a'}_{\ a} = q^{a'}_{\ \nu} q^{\nu}_{\ a} \quad \text{— (21)}$$

Eq. (17) is:

$\partial x^{a'}/\partial x^{a}$ is not zero.

and so
$$q^{a'}_{\ \nu'} = q^{a'}_{\ b'} q^{b'}_{\ \nu'} . \quad \text{— (22)}$$

Eq. (9) is:
$$\Gamma^{a}_{\mu\lambda} = q^{a}_{\ \nu} \Gamma^{\nu}_{\mu\lambda} \quad \text{— (23)}$$

and in eqs. (21) to (23) the tetrad linking the tangent spacetime to the base manifold is always used.