# 209(2): Orbital Dynamics of the Archimedes Spiral.

In this case:

$$r(t) = b\theta(t) \quad -(1)$$

and the angular velocity is:

$$\omega = \frac{d\theta}{dt} = \omega_0 \exp\left(-\frac{\theta^2(t)}{2}\right) \quad -(2)$$

so

$$\int \omega_0 \, dt = \int \exp\left(\frac{\theta^2(t)}{2}\right) d\theta \quad -(3)$$

It is possible to use the result:

$$\int_0^{\theta_1} \exp(\theta^2)\, d\theta = \frac{\sqrt{\pi}}{2} \operatorname{erfi}(\theta_1) \quad -(4)$$

where $\operatorname{erfi}(\theta_1)$ is the imaginary error function. This has the continued fraction expansion:

$$\int_0^{\theta_1} \exp(\theta^2)\, d\theta = \cfrac{\theta_1 \exp(\theta_1^2)}{1 - \cfrac{2\theta_1^2}{3 - \cfrac{4\theta_1^2}{\ddots}}} \quad -(5)$$

It is also possible to use the expansion: $\quad -(6)$

$$\operatorname{erfi}(\theta) = -i + \frac{e^{\theta}}{\sqrt{2}}\left(\frac{1}{\theta} + \frac{1}{2\theta^3} + \cdots\right)$$

In general denote:

$$C_1 + \alpha(\theta(t)) = \int \exp\left(\frac{\theta^2}{2}\right) d\theta \quad -(7)$$

2) where $C_1$ is a constant of integration. Hence:

$$\alpha(\theta(t)) = \omega_0 t + C \quad -(8)$$

from which

$$\theta(t) = \beta(t) \quad -(9)$$

where $\beta$ may be found analytically or numerically. From inspection of the result:

$$\omega_0 t + C_2 = \int \exp\left(\frac{\theta^2(t)}{2}\right) d\theta \quad -(10)$$

the greater is $t$ the greater is $\theta$, so $\theta$ increases with time. This means that the stars emerge from the centre of the galaxy and move outwards.

The torsion is:

$$T^1{}_{01} = \frac{\omega_0}{c}\left(\frac{\theta^2}{1+\theta^2}\right)\exp\left(-\theta^2\right) \quad -(11)$$

where $\theta(t)$ is the function given by eq. (9). The radial coordinate is:

$$r(t) = b\,\theta(t), \quad -(12)$$

the orbital linear velocity is:

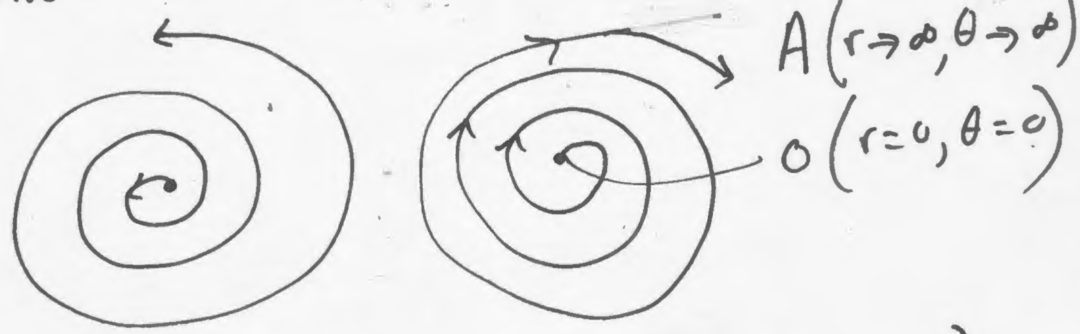
$$v = b\left(1+\theta\right)^{1/2}\omega_0 \exp\left(-\theta^2\right) \quad -(13)$$

and the angular velocity is:

$$\omega = \omega_0 \exp\left(-\frac{\theta^2}{2}\right) \quad -(14)$$

In the case of the Archimedes spiral the

3) radial coordinate r increases with $\theta$, so:




From eq. (13):

$$\frac{dv}{d\theta} = b\omega_0 \exp(-\theta^2) - 2\theta b(1+\theta)^{1/2} e^{-\theta^2}$$

$$= \left(\omega_0 b - 2\theta b(1+\theta)^{1/2}\right) e^{-\theta^2} \quad -(15)$$

Therefore if

$$b \to 0 \quad -(16)$$

then

$$\frac{dv}{d\theta} \to 0. \quad -(17)$$

In this case the outer arm of the spiral behaves like a circular orbit, in which v is constant.

For the hyperbolic spiral:

$$v = \frac{\omega_0 r_0}{\theta^2} + \frac{\omega_0 r_0}{\theta^4} \quad -(18)$$

and v is constant because as $\theta \to \infty$ the spiral becomes a straight line. Unless $\omega_0 r_0$ is very large, the velocity v is very small and directed in a straight line.