209(4): Time Evolution of the General Hyperbolic Spiral.

The general hyperbolic spiral is defined by:

$$r = \frac{r_0}{\theta^n} \quad -(1)$$

so:

$$\frac{dr}{d\theta} = -\frac{r_0 n}{\theta^{n+1}} \quad -(2)$$

$$= -nr/\theta$$

So

$$f = \left(r\frac{d\theta}{dr}\right)^2 = \frac{\theta^2}{n^2}. \quad -(3)$$

Therefore

$$\frac{df}{d\theta} = \frac{2\theta}{n^2}, \quad \frac{d^2f}{d\theta^2} = \frac{2}{n^2} \quad -(4)$$

and

$$F(\theta) = (d^2f/d\theta^2)/(df/d\theta) \quad -(5)$$

$$= \frac{1}{\theta}$$

Therefore for all spirals of this type the equation of motion is:

$$\boxed{\frac{d\omega}{d\theta} = -\frac{\omega}{\theta}} \quad -(6)$$

whose solution is:

$$\theta = \frac{\omega_0}{\omega}. \quad -(7)$$

The general equation for torsion is:

$$T^1_{01} = \frac{df/dt}{c(1+f)} \quad -(8)$$

where

$$\frac{df}{dt} = \frac{1}{2}\frac{df}{dt} = \frac{1}{2}\frac{df}{d\theta}\frac{d\theta}{dt}. \quad -(9)$$

2) The angular velocity is:

$$\omega = \frac{d\theta}{dt} \quad -(10)$$

and therefore:

$$T^{1}_{\ 01} = \frac{\omega}{2c(1+f)} \frac{df}{d\theta} \quad -(11)$$

The torsion for the general spiral of type (1) is therefore:

$$T^{1}_{\ 01} = \frac{1}{cn^2\left(1+\frac{\theta^2}{n^2}\right)}\,\omega \quad -(12)$$

$$\boxed{T^{1}_{\ 01} = \frac{\omega}{c\left(n^2+\theta^2\right)}} \quad -(13)$$

From eqs. (7) and (13):

$$\boxed{T^{1}_{\ 01} = \frac{\omega_0}{c\theta\left(n^2+\theta^2\right)}} \quad -(14)$$

For a given ω_0, and θ the torsion depends on the inverse square of n.

The time dependence of θ is given for all n by:

$$\theta(t) = \sqrt{2}\left(\omega_0 t + C\right)^{1/2} \quad -(15)$$

where C is a constant of integration.

The orbital linear velocity is given by:

3)

$$v = \left(r^2 + \left(\frac{dr}{d\theta} \right)^2 \right)^{1/2} \omega \quad - (16)$$

$$= \left(\frac{r_0^2}{\theta^{2n}} + \frac{n^2 r_0^2}{\left(\theta^{(n+1)}\right)^2} \right) \quad - (17)$$

So:

$$\boxed{v = \frac{r_0^2}{\theta^{2n}} \left(1 + \frac{n^2}{\theta^2} \right)} \quad - (18)$$

where $\theta(t)$ is given by eq. (15). It is seen that:

$$\theta^{n+1} = \theta^n \theta \quad - (19)$$

so

$$\left(\theta^{n+1}\right)^2 = \theta^{2n} \theta^2 \quad - (20)$$

so eq. (18) follows. For large n, then:

$$\frac{dv}{d\theta} \xrightarrow[\theta \to \infty]{} \lim_{\theta \to \infty} \left(-2 r_0^2 n \left(\frac{1}{\theta^{2n+1}} + \frac{n(n+1)}{\theta^{2n+3}} \right) \right) \quad - (21)$$

where $\theta(t)$ is given in eq. (15) and can be expressed in terms of τ as in note 209(3).