206(9): Choice of Definition of the Metric

In curvilinear coordinates the metric in three dimensional space is defined as:

$$g_{ij} = \frac{\partial \underline{r}}{\partial u_i} \cdot \frac{\partial \underline{r}}{\partial u_j} \quad -(1)$$

so in the plane:

$$dZ^2 = 0 \quad -(2)$$

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} . \quad -(3)$$

By definition:

$$d\underline{r} = \frac{\partial \underline{r}}{\partial r} dr + \frac{\partial \underline{r}}{\partial \theta} d\theta \quad -(4)$$

by the chain rule of differentiation. Eq. (4) is an example of the total derivative of differentiation:

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \quad -(5)$$

where

$$u = f(x, y) \quad -(6)$$

and

$$x = x(t),\ y = y(t),\ u = u(t). \quad -(7)$$

(G. Stephenson, "Mathematical Methods for Science Students", Longmans, 1968, page 142.)

So in eq. (4):

$$\underline{r} = f(r, \theta) \quad -(8)$$

$$= r\cos\theta\ \underline{i} + r\sin\theta\ \underline{j}$$

Now consider the constraint function:

$$f = \frac{dr}{d\theta} \quad -(9)$$

For example for the precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)}, \quad -(10)$$

$$\frac{dr}{d\theta} = \left(\frac{\epsilon x}{\alpha}\right) r^2 \sin(x\theta). \quad -(11)$$

Then:

$$d\theta = dr \,/\, f \quad -(12)$$

and:

$$d\underline{r} = \left(\frac{\partial \underline{r}}{\partial r} + \frac{1}{f}\frac{\partial \underline{r}}{\partial \theta}\right) dr \quad -(13)$$

The square of $d\underline{r}$ is:

$$d\underline{r} \cdot d\underline{r} = \left(\left(\frac{\partial \underline{r}}{\partial r} + \frac{1}{f}\frac{\partial \underline{r}}{\partial \theta}\right) \cdot \left(\frac{\partial \underline{r}}{\partial r} + \frac{1}{f}\frac{\partial \underline{r}}{\partial \theta}\right)\right) dr^2$$

$$-(14)$$

$$= \left(\frac{\partial \underline{r}}{\partial r} \cdot \frac{\partial \underline{r}}{\partial r} + \frac{1}{f^2}\frac{\partial \underline{r}}{\partial \theta} \cdot \frac{\partial \underline{r}}{\partial \theta} + \frac{1}{f}\left(\frac{\partial \underline{r}}{\partial r} \cdot \frac{\partial \underline{r}}{\partial \theta} + \frac{\partial \underline{r}}{\partial \theta} \cdot \frac{\partial \underline{r}}{\partial r}\right)\right) dr^2$$

The two metric elements for eq. (1) are:

$$g_{11} = \frac{\partial \underline{r}}{\partial r} \cdot \frac{\partial \underline{r}}{\partial r} \quad -(15)$$

$$g_{22} = \frac{\partial \underline{r}}{\partial \theta} \cdot \frac{\partial \underline{r}}{\partial \theta} \quad -(16)$$

3) From eq. (13)

$$\boxed{\frac{d\underline{r}}{dr} = \frac{\partial \underline{r}}{\partial r} + \frac{1}{f}\frac{\partial \underline{r}}{\partial \theta}} \quad -(17)$$

so the total derivative is the sum of two partial derivatives.

Now define the unit vectors:

$$\underline{e}_r = \cos\theta \, \underline{i} + \sin\theta \, \underline{j} \quad -(18)$$

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(19)$$

then:

$$\frac{d\underline{r}}{dr} = \frac{\partial \underline{r}}{\partial r} + \frac{1}{f}\frac{\partial \underline{r}}{\partial \theta}$$

$$= \cos\theta \, \underline{i} + \sin\theta \, \underline{j} + \frac{r}{f}\left(-\sin\theta \, \underline{i} + \cos\theta \, \underline{j}\right)$$

$$\boxed{\frac{d\underline{r}}{dr} = \underline{i}\left(\cos\theta - \frac{r}{f}\sin\theta\right) + \underline{j}\left(\sin\theta + \frac{r}{f}\cos\theta\right)} \quad -(18)$$

Finally define the metric as:

$$\boxed{g_{11} = \frac{d\underline{r}}{dr} \cdot \frac{d\underline{r}}{dr}} \quad -(19)$$

Instead of defining the metric as consisting of two elements, as in eq. (1), it is defined as consisting of one element. The one element is defined using the total derivative, eq. (17).

4) This procedure is allowed because the metric is a matter of definition.

The advantage of eq. (19) is that it reduces the theory to its simplest form by Ockham's Razor. From eqs. (18) and (19):

$$\boxed{g_{11} = \left(1 + \frac{r^2}{f^2}\right)} \quad -(20)$$

which is the definition used in UFT 205.

The complete spacetime metric is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -\left(1 + \frac{r^2}{f^2}\right) \end{bmatrix} \quad -(21)$$

again as used in UFT 205.