206(5): First Equation of Motion from Constrained Metric.

Consider the constrained metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - (x^2 + r^2)\, d\theta^2 \quad -(1)$$

where

$$x = dr / d\theta. \quad -(2)$$

The Lagrangian is: $\mathcal{L} = \frac{1}{2} mc^2 \quad -(3)$

and

$$mc^2 = mc^2 \left(\frac{dt}{d\tau}\right)^2 - m(x^2 + r^2)\, d\theta^2 \quad -(4)$$

The Euler Lagrange equation gives:

$$E = mc^2 \frac{dt}{d\tau}, \quad L = m(x^2 + r^2) \frac{d\theta}{d\tau} \quad -(5)$$

where E is the energy and L the angular momentum.

So:

$$mc^2 = \frac{E^2}{mc^2} - \frac{L^2}{m(x^2 + r^2)} \quad -(6)$$

i.e.

$$mc^2 = \frac{E^2}{mc^2} - \frac{L^2}{m\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)} \quad -(7)$$

Therefore:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{L^2}{m^2\left(\frac{E^2}{mc^2} - mc^2\right)} - r^2, \quad -(8)$$

which gives the equation:

2)

$$\boxed{\frac{c^2 L^2}{m\left(E^2 - m^2 c^4\right)} = r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad -(9)$$

In the limit of free particle motion:

$$\frac{dr}{d\theta} \rightarrow 0 \quad -(10)$$

and

$$c^2 L^2 \rightarrow m r^2 \left(E^2 - m^2 c^4\right) \quad -(11)$$

For a free particle in linear motion:

$$E^2 - m^2 c^4 \rightarrow p^2 c^2 \quad -(12)$$

where p is the relativistic momentum:

$$p = \gamma m v = \gamma m \frac{dr}{dt} \quad -(13)$$

In this case:

$$L^2 \rightarrow m p^2 r^2 \quad -(14)$$

In this limit the angular momentum and linear momentum become non-relativistic, so:

$$L \rightarrow m v r \quad -(15)$$

which is the non-relativistic result self-consistently, QED.

If we consider in eq. (9):

$$E^2 - m^2 c^4 \rightarrow p^2 c^2 \quad -(16)$$

3) Then the equation becomes:

$$\boxed{L^2 \rightarrow m r^2 p^2 \left(1 + \left(\frac{dr}{d\theta} \right)^2 \right)} \quad -(17)$$

For an elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(18)$$

then

$$L^2 \rightarrow m r^2 p^2 \left(1 + \left(\frac{\epsilon}{\alpha} \right)^2 r^2 \sin^2\theta \right) \quad -(19)$$

and for a precessing elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(20)$$

$$L^2 \rightarrow m r^2 p^2 \left(1 + \left(\frac{x\epsilon}{\alpha} \right)^2 r^2 \sin^2\theta \right) \quad -(21)$$

By definition:

$$c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(22)$$

where

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2 = (x^2 + r^2)\, d\theta^2 \quad -(23)$$

so

$$c^2 d\tau^2 = (c^2 - v^2)\, dt^2 \quad -(24)$$

and

$$d\tau^2 = \left(1 - \frac{v^2}{c^2} \right) dt^2 \quad -(25)$$

So:

$$\frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (26)$$

where:

$$v = (x^2 + r^2)^{1/2} \frac{d\theta}{dt} \quad - (27)$$

In this theory the angular momentum is:

$$L = m(x^2 + r^2) \frac{d\theta}{d\tau} \quad - (28)$$

$$= m(x^2 + r^2) \frac{d\theta}{dt} \frac{dt}{d\tau}$$

From eq. (27):

$$x^2 + r^2 = \frac{v^2}{\omega^2} \quad - (29)$$

where

$$\omega = \frac{d\theta}{dt} \quad - (30)$$

is the angular velocity.

So:

$$L = \gamma \frac{mv^2}{\omega} \quad - (31)$$

where

$$\gamma = dt / d\tau, \quad - (32)$$

i.e

$$\boxed{\omega L = \gamma m v^2} \quad - (33)$$

In the limit:

)

$$v << c \quad -(34)$$

then

$$\omega L = m v^2 . \quad -(35)$$

which is the non-relativistic result, QED.

From eq. (28):

$$\boxed{L = \gamma m \left(r^2 + \left(\frac{dr}{d\theta}\right)^2 \right) \omega} \quad -(36)$$

In the limit:

$$\frac{dr}{d\theta} \rightarrow 0, \quad \gamma \rightarrow 1 \quad -(37)$$

this becomes the Newtonian result:

$$L = m r^2 \omega . \quad -(38)$$

Note carefully that in this theory, the angular momentum L is not a constant of motion, because the lagrangian is defined by eq. (3). It becomes a constant of motion in the Newtonian limit, self consistently.

<u>Conclusion</u>

The constrained metric (1) is rigorously self consistent.