206(6): Second Derivation of the Equation of Motion of the Constrained Minkowski Metric.

Consider the constrained metric:

$$ds^2 = c^2 dt^2 - \left(1 + \left(\frac{r\,d\theta}{dr}\right)^2\right) dr^2 \quad -(1)$$

for which the lagrangian is:

$$\mathcal{L} = \frac{1}{2} mc^2 = \frac{1}{2} mc^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2} m \left(1 + \left(\frac{r\,d\theta}{dr}\right)^2\right)\left(\frac{dr}{d\tau}\right)^2. \quad -(2)$$

The Euler Lagrange equation gives:

$$E = mc^2 \left(\frac{dt}{d\tau}\right), \quad -(3)$$

which is the total energy.

So:

$$mc^2 = \frac{E^2}{mc^2} - m\left(1 + \left(\frac{r\,d\theta}{dr}\right)^2\right)\left(\frac{dr}{d\tau}\right)^2. \quad -(4)$$

Now use:

$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\frac{d\theta}{d\tau}, \quad -(5)$$

so:

$$\frac{E^2}{mc^2} - mc^2 = m\left(1 + r^2\left(\frac{d\theta}{dr}\right)^2\right)\left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{d\tau}\right)^2 \quad -(6)$$

$$= m\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\left(\frac{d\theta}{d\tau}\right)^2.$$

From eq. (7) of note 206(5):

2)

$$\frac{E^2}{mc^2} - mc^2 = \frac{L^2}{m\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)} \quad -(7)$$

Comparing eqs. (6) and (7):

$$L = m\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\frac{d\theta}{d\tau} \quad -(8)$$

which is the same result as in note for (5).

So the two methods of constraint produce the same equation:

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 = \frac{c^2L^2}{m(E^2 - m^2c^4)} - r^2} \quad -(9).$$

which expresses any orbit in terms of the total energy:

$$\boxed{E = \gamma mc^2} \quad -(10)$$

and the total angular momentum:

$$L = m\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\frac{d\theta}{d\tau}$$

$$\boxed{L = \gamma m\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\frac{d\theta}{dt}.} \quad -(11)$$

If the total velocity is v, then:

$$\omega L = \gamma m v^2 \quad - (12)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (13)$$

and

$$v = \left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{1/2} \frac{d\theta}{dt} \quad - (14)$$

The angular velocity is:

$$\omega = \frac{d\theta}{dt}, \quad - (15)$$

so

$$\boxed{v = \left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{1/2} \omega} \quad - (16).$$

<u>for any orbit.</u>

From eq. (14):

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 + r^2 = \left(\frac{v}{\omega}\right)^2} \quad - (17)$$

so

$$\boxed{\left(\frac{v}{\omega}\right)^2 = \frac{c^2 L^2}{m\left(E^2 - m^2 c^4\right)}} \quad - (18)$$

In the Newtonian limit:

$$\gamma \to 1 \quad - (19)$$

4)

and
$$E^2 - m^2c^4 \rightarrow c^2p^2 \quad -(20)$$

where
$$p = \gamma m v \rightarrow m v. \quad -(21)$$

So from eq. (12):
$$\omega L \rightarrow m v^2, \quad -(22)$$

therefore
$$\frac{c^2L^2}{m(E^2 - m^2c^4)} \rightarrow \left(\frac{v}{\omega}\right)^2 \quad -(23)$$

Q.E.D.

The set of equations is again rigorously self-consistent.

Eq. (16) appears to be a completely new equation of orbits. For a circle:

$$\frac{dr}{d\theta} = 0 \quad -(24)$$

so
$$v = \omega r. \quad -(25)$$

For an ellipse:
$$v = \left(\left(\frac{\epsilon}{\alpha}\right)^2 r^4 \sin^2\theta + r^2\right)^{1/2} \omega \quad -(26)$$

Note that eq. (16) is:

$$v^2 = \omega^2\left(\frac{dr}{d\theta}\right)^2 + r^2\omega^2$$

$$= \left(\frac{d\theta}{dt}\right)^2\left(\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(27)$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

which is the expression for v^2 obtained by differentiating

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d(r\,\underline{e}_r)}{dt} \quad -(28)$$

in cylindrical polar coordinates. QED.

Therefore eq. (27) is obtained directly from the constrained metric:

$$c^2 d\tau^2 = c^2 dt^2 - d\underline{r}\cdot d\underline{r}$$

$$= v^2 dt^2 \quad -(29)$$

$$= \left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) d\theta^2$$

In the Newtonian theory the total velocity v is defined by:

$$E_N = \frac{1}{2}mv^2 - \frac{mMG}{r} \quad -(30)$$

so

$$v = \left(\frac{2}{m}\left(E_N + \frac{mMG}{r}\right)\right)^{1/2} \quad -(31)$$

and:

6)

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) \quad -(32)$$

The angular momentum of the Newtonian theory is defined by:

$$L \rightarrow mr^2\omega \quad -(33)$$

which is the limit of eq. (11) when:

$$\gamma \rightarrow 1, \quad \frac{dr}{d\theta} \ll r. \quad -(34)$$

The total kinetic energy of the Newtonian theory is:

$$T = \gamma mc^2 - mc^2 \rightarrow \frac{1}{2}mv^2 \quad -(35)$$

In the Newtonian limit, eq. (9) becomes eq. (27) as:

$$\frac{c^2L^2}{m(E^2 - m^2c^4)} \longrightarrow \left(\frac{v}{\omega}\right)^2 \quad -(36)$$

So conversely the Newtonian theory is generalized by:

$$v \longrightarrow \omega\left(\frac{c^2L^2}{m(E^2-m^2c^4)}\right)^{1/2} \quad -(37)$$

From eqs. (27) and (37):

$$\boxed{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \omega^2\left(\frac{c^2L^2}{m(E^2-m^2c^4)}\right)} \quad -(38)$$

7) The Newtonian point of view is related to eq. (38) by:

$$\boxed{v^2 = \frac{2}{m}\left(E_N + \frac{mMG}{r}\right) \longrightarrow \omega^2\left(\frac{c^2L^2}{m(E^2 - m^2c^4)}\right) \quad -(39)}$$

which means that the concept of potential energy of attraction is replaced by the concept of a constrained metric.

In the Newtonian theory the ellipse is given by eq. (17) as follows:

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{v}{\omega}\right)^2 - r^2; \quad -(40)$$

$$= \frac{2}{m\omega^2}\left(E_N + \frac{mMG}{r}\right) - r^2$$

$$= \left(\frac{\epsilon}{\alpha}\right)^2 r^4 \sin^2\theta \quad -(41)$$

and in the constrained metric theory:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{c^2L^2}{m(E^2 - m^2c^4)} - r^2 \quad -(42)$$

$$= \left(\frac{\epsilon}{\alpha}\right)^2 r^4 \sin^2\theta$$