201(7): Proof that the Inverse Square Law is Not Unique

Consider the Newtonian theory of orbits in which the total energy is:

$$E = T + V \quad -(1)$$

and is a constant. Here T is the kinetic energy:

$$T = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) \quad -(2)$$

for an orbit in the plane:

$$dZ^2 = 0, \quad -(3)$$

and V is the potential energy. It is almost always claimed in the dogma that:

$$V = -\frac{k}{r} \quad -(4)$$

where k is a constant:

$$k = mMG. \quad -(5)$$

Here M is the attracting mass, m the attracted mass and G is Newton's constant.

In this note it is shown that the choice (4) is not unique, there is no universal force law.

Proof: From eqs. (1) and (2):

$$E = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} m r^2 \left(\frac{d\theta}{dt} \right)^2 + V(r) \quad -(6)$$

a)

The Newtonian angular momentum L is constant and is:

$$L = mr^2 \frac{d\theta}{dt} \quad -(7)$$

So eq. (6) is:

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} + V(r) \quad -(8)$$

Therefore:

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m}\left(E - \frac{L^2}{2mr^2} - V(r)\right) \quad -(9)$$

Using $$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \omega \frac{dr}{d\theta} \quad -(10)$$

it is found that:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2}{m\omega^2}\left(E - \frac{L^2}{2mr^2} - V(r)\right)$$

$$= \frac{2mr^4}{L^2}\left(E - \frac{L^2}{2mr^2} - V(r)\right) \quad -(11)$$

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 = \frac{2mr^4\left(E - V(r)\right)}{L^2} - r^2} \quad -(12)$$

By <u>observation</u> the orbit of a planet is solar system is an ellipse, to an excellent approximation. This was first inferred by Johannes Kepler.

Thus:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(13)$$

where (r, θ) are the cylindrical polar coordinates in the plane (3). More generally eq. (13) is that of a conic section. For the ellipse, 2α is its right magnitude (latus rectum) and ϵ its eccentricity. These are constants of the ellipse.

From eq. (13):

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{\epsilon}{\alpha}\right)^2 r^4 \sin^2\theta \quad -(14)$$

$$= \left(\frac{\epsilon}{\alpha}\right)^2 r^4 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)$$

Comparing eqns. (11) and (14):

$$E - V(r) - \frac{L^2}{2mr^2} = \left(\frac{\epsilon}{\alpha}\right)^2 \frac{L^2}{2m}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) \quad -(15)$$

from which:

$$\boxed{V(r) = E - \frac{L^2}{2m}\left(\frac{2}{\alpha r} + \left(\frac{\epsilon^2 - 1}{\alpha^2}\right)\right)} \quad -(16)$$

in which E, L, α and ϵ are constants, only two of which (α and ϵ) can be determined by observation.

So it is not possible to determine $V(r)$ by observation, QED.

In the dogma of three centuries, the choice is made of:

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2}. \quad -(17)$$

From eqs. (17) and (16):

$$V(r) = -\frac{k}{r} = -\frac{mMG}{r}. \quad -(18)$$

However, for an observed α and ϵ, $V(r)$ can be determined only up to E and L in eq. (16). The choice (18) is not unique, there is no universal law of gravitation.

The kinetic energy of the Newtonian theory is given from eq. (16) as:

$$T = E - V(r) = \frac{L^2}{2m}\left(\frac{2}{\alpha r} + \left(\frac{\epsilon^2 - 1}{\alpha^2}\right)\right)$$

$$= \frac{1}{2}mv^2 \quad -(19)$$

So the total linear velocity of m is:

$$v^2 = L^2 \left(\frac{2}{\alpha r} + \left(\frac{\epsilon^2 - 1}{\alpha^2} \right) \right) \quad -(20)$$

The only quantity that can be determined experimentally is:

$$\left(\frac{v}{L} \right)^2 = \frac{2}{\alpha r} + \left(\frac{\epsilon^2 - 1}{\alpha^2} \right) \quad -(21)$$

Now denote the Newtonian total energy as E_N and the Newtonian angular momentum as L_N for clarity. In the Newtonian theory these are defined as constants. In the more general constrained metric theory:

$$\left(\frac{dr}{d\theta} \right)^2 = \frac{c^2 L^2}{E^2 - m^2 c^4} - r^2 \quad -(22)$$

in which the only constant is:

$$\mathcal{L} = \frac{1}{2} mc^2 \quad -(23)$$

In the general theory (22) there is no potential energy, a property of general relativity.

The reduction to Newton therefore occurs by artificially constraining E and L to be constants, and by artificially introducing $V(r)$:

$$\left(\frac{v}{\omega}\right)^2 = \frac{c^2 L^2}{E^2 - m^2 c^4} \longrightarrow \frac{2mr^4\left(E_N - V(r)\right)}{L_N^2} \qquad -(24)$$

<u>The Newtonian theory</u> is <u>only</u> one <u>out of an infinite possibilities.</u> It is therefore entirely wrong to claim that it predicts an elliptical orbit.

The only thing that can be said is that an orbit can be <u>analysed</u> by:

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 = \frac{v^2}{\omega^2} - r^2} \qquad -(25)$$
