206(3): Development of the Equations of the Identity.

As in note 205(4) the equations of the identity relevant to any orbit in a plane are:

$$D_0 T^1{}_{01} := R^1{}_{001} \quad -(1)$$

and

$$D_2 T^1{}_{21} := R^1{}_{221} \quad -(2)$$

In ECE philosophy they are field equations of dynamics. Their structure is deduced from a Minkowski metric of special relativity constrained by an orbit.

As shown in note 205(4) the equations reduce to:

$$\frac{d^2 f}{dt^2} = \frac{d^2 f}{d\theta^2}\left(\frac{d\theta}{dt}\right)^2 \quad -(3)$$

where

$$f = \left(r\frac{d\theta}{dr}\right)^2. \quad -(4)$$

The constrained Minkowski metric produces elements of torsion and curvature which are related by the Evans identity of differential geometry.

The identity reduces to eq. (3), which gives the time evolution of f. Eq. (3) is similar to a wave equation or diffusion equation.

In order to check eq. (3) and develop it recall some elements of differentiation as follows (G. Stephenson, "Mathematical Methods for Science Students" (Longmans, 1968)).

The derivative of f with respect to x is:

2)

$$\frac{df}{dx} = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right) \quad -(5)$$

In abbreviated notation the Leibnitz theorem is:

$$D(uv) = u\,Dv + v\,Du \quad -(6)$$

i.e.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad -(7)$$

For a function of two variable $f(x,y)$, the first partial derivative is:

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x, y) - f(x,y)}{\delta x} \right) \quad -(8)$$

If f is a function of u and u is a function of x, then:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \quad -(9)$$

If f is a function of u and u is a function of x and y then:

$$\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} \quad -(10)$$

Consider a function:

$$g = g(u(x,y)) \quad -(11)$$

then from eq. (10):

$$\frac{\partial g}{\partial x} = \frac{dg}{du}\frac{\partial u}{\partial x} \quad -(12)$$

Now let:
$$g = \frac{\partial f}{\partial u} \quad -(13)$$
then:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \frac{d^2 f}{du^2}\frac{\partial u}{\partial x} \quad -(14)$$
i.e.
$$\boxed{\frac{d^2 f}{du^2} = \frac{\partial x}{\partial u}\frac{\partial^2 f}{\partial x\, \partial u}} \quad -(15)$$
Secondly let:
$$g = \frac{\partial f}{\partial x} \quad -(16)$$
then
$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = \frac{d^2 f}{dx^2}\frac{\partial x}{\partial u} \quad -(17)$$
i.e.
$$\boxed{\frac{d^2 f}{dx^2} = \frac{\partial u}{\partial x}\frac{\partial^2 f}{\partial u\, \partial x}} \quad -(18)$$
As shown by Stephenson on page 139, the operators $\partial / \partial u$ and $\partial / \partial x$ are commutative:
$$\frac{\partial^2 f}{\partial u\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial u} \quad -(19)$$
if such functions are continuous at (a, b).

Dividing eq. (15) by eq. (18):
$$\boxed{\left(\frac{\partial x}{\partial u}\right)^2 = \frac{d^2 f}{du^2} \Big/ \frac{d^2 f}{dx^2}} \quad -(20)$$

and

$$\frac{d^2 f}{du^2} = \left(\frac{\partial x}{\partial u}\right)^2 \frac{d^2 f}{dx^2} \quad -(21)$$

If $u = t$, $x = \theta$ then eq. (3) follows. So eqs. (1) and (2) are exact identities of differential geometry that lead to the rules of differentiation. Eq. (3) is an exact time evolution equation and these results are true for all rotational motion in a plane.

Eq. (11) becomes:

$$g = g(t(\theta, r)) \quad -(22)$$

which means that g is a function of t which is a function of θ and r.

The total derivative is defined by:

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y) \quad -(23)$$

so:

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \quad -(24)$$

where:

$$u = f(x, y) \quad -(25)$$

and

$$x = x(t), \; y = y(t). \quad -(26)$$

If:

$$t = x \quad -(27)$$

then:

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad -(28)$$

i.e.

$$\boxed{\frac{df(x,y)}{dx} = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}} \quad -(29)$$

If

$$l = \frac{dx}{dt}, \quad k = \frac{dy}{dt} \quad -(30)$$

then:

$$\frac{d^2u}{dt^2} = l^2\frac{\partial^2 f}{\partial x^2} + 2lk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \quad -(31)$$

For example, if

$$f(x,y) = \theta(t, r) \quad -(32)$$

i.e.

$$x = t, \; y = r \quad -(33)$$

then:

$$\boxed{\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial r}\frac{dr}{dt}} \quad -(34)$$

The angular velocity is defined as:

$$\omega = \frac{d\theta}{dt} \quad -(35)$$

and is the total derivative of θ with respect to time t.

The overall method is based on the

Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(36)$$

in which:

$$\frac{dr}{d\theta} = g(r, \theta). \quad -(37)$$

So:

$$\boxed{ds^2 = c^2 dt^2 - dr^2 (1 + f)} \quad -(38)$$

where

$$f = \left(r \frac{d\theta}{dr} \right)^2. \quad -(39)$$

There are two metric elements:

$$g_{00} = 1, \quad g_{11} = -(1+f). \quad -(40)$$

The antisymmetric connection is found from the ~~theorem~~ of metric compatibility:

$$D_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda}$$
$$= 0 \quad -(41)$$

which gives:

$$\Gamma^1_{01} = \frac{1}{2} g_{11}^{-1} \partial_0 g_{11} \quad -(42)$$

$$= \frac{1}{2c} g_{11}^{-1} \frac{\partial g_{11}}{\partial t}$$

$$\boxed{\Gamma^1_{01} = \frac{1}{2c} \frac{\partial f / \partial t}{2(1+f)}} \quad -(43)$$

and:

$$\Gamma^1_{21} = \frac{1}{2g_{11}} \partial_2 g_{11} \quad - (44)$$

in which

$$\partial_2 g_{11} = \frac{1}{2r} \frac{\partial g_{11}}{\partial \theta} \quad - (45)$$

in cylindrical polar coordinates. So

$$\boxed{\Gamma^1_{21} = \frac{1}{2r(1+f)} \frac{\partial f}{\partial \theta}} \quad - (46)$$

Thus:

$$T^1_{01} = 2\Gamma^1_{01} \quad - (47)$$

$$T^1_{21} = 2\Gamma^1_{21} \quad - (48)$$

These are two non-zero torsion elements. Computer algebra is applied to find the non-zero curvature elements, which is these are related by the Evans identity, an example of the Cartan identity of differential geometry:

$$\boxed{D_\mu T^{\kappa\mu\nu} := R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad - (49)$$

which gives one half of the field equations. The other half is given by the Cartan identity:

$$\boxed{D_\mu \tilde{T}^{\kappa\mu\nu} := \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad - (50)$$

where the tilde denotes Hodge dual. Eqs.

8) (49) and (50) are interconvertible because the Hodge dual of a rank two tensor in μ and ν is another rank two tensor in μ and ν in four dimensions. Here κ is field constant.

To raise indices, the metric is used:

$$T^{\kappa\mu\nu} = g^{\mu\alpha} g^{\nu\beta} T^{\kappa}{}_{\alpha\beta} \quad -(51)$$

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R^{\kappa}{}_{\mu\alpha\beta} . \quad -(52)$$

The rule for covariant differentiation of a rank three tensor is:

$$D_\mu T^{\kappa}{}_{\nu\sigma} = \partial_\mu T^{\kappa}{}_{\nu\sigma} + \Gamma^{\kappa}_{\mu\lambda} T^{\lambda}{}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} T^{\kappa}{}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}{}_{\nu\lambda} . \quad -(53)$$

The relevant curvature elements are:

$$R^{1}{}_{001} = \frac{1}{c^2}\left(\frac{2(1+f)\,\partial^2 f/\partial t^2 - (\partial f/\partial t)^2}{4(1+f)^2}\right) \quad -(54)$$

and

$$R^{1}{}_{221} = \frac{1}{r^2}\left(\frac{2(1+f)\,\partial^2 f/\partial \theta^2 - (\partial f/\partial \theta)^2}{4(1+f)^2}\right) \quad -(55)$$

<u>Partial derivatives</u> <u>are used</u> in <u>the</u> <u>definition</u> of <u>torsion</u> and <u>curvature</u>.

9) In order to evaluate these, recall that f is a function of r, θ and t:

$$f = r^2 \left(\frac{d\theta}{dx}\right)^2 = f(r, \theta, t) \quad - (56)$$

so:

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t}} \quad - (57)$$

which is an application of eq. (10) in which f of that equation is a function of more than one variable.

The computer algebra gives:

$$6 \frac{\partial^2 f}{\partial t^2}(1+f) = 5\left(\frac{\partial f}{\partial t}\right)^2 \quad - (58)$$

$$6 \frac{\partial^2 f}{\partial \theta^2}(1+f) = 5\left(\frac{\partial f}{\partial \theta}\right)^2 \quad - (59)$$

Partial derivatives also appear in these equations.

In general, these equations are differential equations which must be solved simultaneously. They have an obvious symmetry of structure.

Dividing eq. (58) by eq. (59) gives the following exact identity:

10)

$$\boxed{\frac{\partial^2 f}{\partial t^2} := \frac{\partial^2 f}{\partial \theta^2}\left(\frac{\partial \theta}{\partial t}\right)^2.} \quad -(60)$$

which gives the time evolution of f.

From eq. (34):

$$\boxed{\begin{aligned}\omega &= \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial r}\frac{dr}{dt} \\ &= \frac{d\theta}{dt}\end{aligned}} \quad -(61)$$

Eq. (60) is given by differential algebra as follows:

$$g = \frac{\partial f}{\partial t}, \quad \frac{\partial g}{\partial t} = \frac{\partial^2 f}{\partial t^2}. \quad -(62)$$

Use:

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial t} = \left(\frac{\partial^2 f}{\partial \theta\,\partial t}\right)\left(\frac{\partial \theta}{\partial t}\right). \quad -(63)$$

Now use:

$$h = \frac{\partial f}{\partial \theta}, \quad \frac{\partial h}{\partial \theta} = \frac{\partial^2 f}{\partial \theta^2}, \quad -(64)$$

and

$$\frac{\partial h}{\partial \theta} = \frac{\partial h}{\partial t}\frac{\partial t}{\partial \theta} = \left(\frac{\partial^2 f}{\partial t\,\partial \theta}\right)\left(\frac{\partial t}{\partial \theta}\right) \quad -(65)$$

By isotropy:

$$\frac{\partial^2 f}{\partial \theta\,\partial t} = \frac{\partial^2 f}{\partial t\,\partial \theta} \quad -(66)$$

11) We have:

$$\frac{\partial^2 f}{\partial t^2} = \left(\frac{\partial^2 f}{\partial\theta\,\partial t}\right)\left(\frac{\partial\theta}{\partial t}\right) \quad -(67)$$

$$\frac{\partial^2 f}{\partial\theta^2} = \left(\frac{\partial^2 f}{\partial t\,\partial\theta}\right)\left(\frac{\partial t}{\partial\theta}\right). \quad -(68)$$

Dividing gives eq. (60), QED.

Note that partial derivatives are used throughout. The angular velocity is the total derivative:

$$\omega = \frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial r}\frac{dr}{dt} \quad -(69)$$

and eq. (60) is a new general equation of orbits that is also an equation of differential algebra.

If: $$\frac{\partial\theta}{\partial t} \sim \text{constant} \quad -(70)$$

then eq. (60) is a diffusion equation or wave equation.
