206(8): Orbital Equation for Unconstrained Metric

In this case:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

and the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m c^2 = \frac{1}{2} m \left(c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right) \quad -(2)$$

So:

$$mc^2 = \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{mr^2} \quad -(3)$$

where

$$E = \gamma m c^2, \quad L = m r^2 \left(\frac{d\theta}{d\tau}\right). \quad -(4)$$

So:

$$m\left(\frac{dr}{d\tau}\right)^2 = m\left(\frac{dr}{d\theta}\right)^2 \left(\frac{d\theta}{d\tau}\right)^2 = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} \quad -(5)$$

i.e. from eqs. (4) and (5):

$$\left(\frac{dr}{d\theta}\right)^2 = r^4 \left(\frac{1}{b^2} - \frac{1}{a^2} - \frac{1}{r^2} \right) \quad -(6)$$

where:

$$b = \frac{Lc}{E}, \quad a = \frac{mc}{L}. \quad -(7)$$

So

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{E^2 - m^2c^4}{c^2 L^2}\right) r^4 - r^2} \quad -(8)$$

The constrained metric gives:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{c^2 L_c^2}{E^2 - m^2 c^4} - r^2 \quad -(9)$$

where

$$L_c = m\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\frac{d\theta}{d\tau} \quad -(10)$$

From eqs (8) and (10):

$$\boxed{L_c = \left(\frac{E^2 - m^2c^4}{c^2 L}\right)r^2} \quad -(11)$$

However, for the unconstrained metric:

$$\frac{dr}{d\theta} = 0 \quad -(12)$$

because this is the case of special relativity where the mass m moves in a straight line. So:

$$\frac{E^2 - m^2c^4}{c^2L^2} = \frac{1}{r^2} \quad -(13)$$

from eq. (8), i.e.

$$E^2 - m^2c^4 = c^2\left(\frac{L}{r}\right)^2 = c^2p^2 \quad -(14)$$

where

$$p = \frac{L}{r} \quad -(15)$$

and

$$p = \gamma m v, \quad L = \gamma m v r. \quad -(16)$$

So

$$E^2 = c^2p^2 + m^2c^4 \quad -(17)$$

which is a re-expression of:

$$p = \gamma m v \quad -(18)$$

3) These are self-consistently the results of special relativity. The constraint introduces an orbit, and reduces the dimensionality of the line element.

On the other hand the incorrect Schwarzschild metric gives:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{1}{b^2} - \left(1 - \frac{r_0}{r}\right)\frac{1}{a^2}\right) - \left(1 - \frac{r_0}{r}\right)r^2 \quad - (19)$$

and this is not compatible with the correct result:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{c^2 L_c^2}{E^2 - m^2 c^4} - r^2 \quad - (20)$$