205(3): Orbital Torsion for Spirals

The two relevant elements are:

$$T^1{}_{01} = \frac{1}{c}\left(1+f\right)^{-1}\frac{df}{dt} \quad -(1)$$

and

$$T^1{}_{12} = \frac{1}{r}\left(1+f\right)^{-1}\frac{df}{d\theta} \quad -(2)$$

$$= \frac{1}{r}\left(1+f\right)^{-1}\frac{df}{dt}\frac{dt}{d\theta}$$

with the correct units of inverse metres. Here:

$$f = r^2\left(\frac{d\theta}{dr}\right)^2 \quad -(3)$$

These equations are true for all orbits.

Logarithmic Spiral

Here

$$r = r_0 e^{\alpha\theta} \quad -(4)$$

so

$$f = \alpha^2 \quad -(5)$$

and

$$\boxed{T^1{}_{01} = T^1{}_{12} = 0} \quad -(6)$$

Hyperbolic Spiral

Here

$$r = \frac{r_0}{\theta} \quad -(7)$$

and

$$f = \left(\frac{r_0}{r}\right)^2. \quad -(8)$$

So:

2)

$$T^{1}_{01} = \frac{1}{c}\left(1+\left(\frac{r_0}{r}\right)^2\right)^{-1}\frac{df}{dt}$$

$$= \frac{1}{c}\left(1+\theta^2\right)^{-1}\frac{df}{d\theta}\frac{d\theta}{dt} \quad -(9)$$

Here:

$$f = \left(\frac{r_0}{r}\right)^2 = \theta^2 \quad -(10)$$

$$\frac{df}{d\theta} = 2\theta \quad -(11)$$

So:

$$\boxed{T^{1}_{01} = \frac{1}{c}\left(\frac{2\theta}{1+\theta^2}\right)\frac{d\theta}{dt}} \quad -(12)$$

and

$$\boxed{T^{1}_{12} = \frac{2\theta}{r(1+\theta^2)}} \quad -(13)$$

The factor r in the denominator of eq. (13) comes from the definition of the divergence in cylindrical coordinates:

$$\underline{\nabla}\cdot\underline{F} = \frac{1}{r}\left(\frac{\partial(rF_r)}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right) \quad -(14)$$

in the plane

$$dZ^2 = 0 \quad -(15)$$
