205(6): The General Curvilinear Coordinates

Consider the position vector $\underline{r}$ of a point P in three dimensional space:

$$\underset{\sim}{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} \quad - (1)$$

in Cartesian coordinates. Consider the curvilinear coordinate system (u_1, u_2, u_3). Then:

$$- (2)$$

$$X = X(u_1, u_2, u_3) \quad - (3)$$

$$Y = Y(u_1, u_2, u_3) \quad - (4)$$

$$Z = Z(u_1, u_2, u_3)$$

$$- (5)$$

and

$$\underset{\sim}{r} = \underline{r}(u_1, u_2, u_3)$$

The unit vectors of the curvilinear system are:

$$\underline{e}_1 = \frac{1}{h_1}\frac{\partial \underline{r}}{\partial u_1}, \quad \underline{e}_2 = \frac{1}{h_2}\frac{\partial \underline{r}}{\partial u_2}, \quad \underline{e}_3 = \frac{1}{h_3}\frac{\partial \underline{r}}{\partial u_3} \quad - (6)$$

The scale factors are:

$$h_i = \left| \frac{\partial \underline{r}}{\partial u_i} \right| \quad - (7)$$

The del operator is:

$$\underline{\nabla} = \frac{\underline{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\underline{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\underline{e}_3}{h_3}\frac{\partial}{\partial u_3} \quad - (8)$$

The infinitesimal line element is

$$d\underline{r} \cdot d\underline{r} = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2 \quad - (9)$$

The metric is:

$$g_{ij} = \frac{\partial \underline{r}}{\partial u_i} \cdot \frac{\partial \underline{r}}{\partial u_j} \quad -(10)$$

If: $g_{ij} = 0$ for $i \neq j$ $\quad -(11)$

the coordinate system is orthogonal and

$$g_{11} = h_1^2,\ g_{22} = h_2^2,\ g_{33} = h_3^2 \quad -(12)$$

The Minkowski metric in curvilinear coordinates is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -h_1^2 & 0 & 0 \\ 0 & 0 & -h_2^2 & 0 \\ 0 & 0 & 0 & -h_3^2 \end{bmatrix} \quad -(13)$$

and:

$$ds^2 = c^2 dt^2 - h_1^2 du_1^2 - h_2^2 du_2^2 - h_3^2 du_3^2 . \quad -(14)$$

The constraints are:

$$\frac{du_1}{du_2} = f(u_1, u_2) \quad -(15)$$

and so on.

The gradient of a function f in orthogonal curvilinear coordinates is:

$$\underline{\nabla} f = f_1 \underline{e}_1 + f_2 \underline{e}_2 + f_3 \underline{e}_3 \quad -(16)$$

By definition:

$$df = \underline{\nabla} f \cdot d\underline{r}, \quad (17)$$

where:

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3 \quad -(18)$$

and:

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3 \quad -(19)$$

$$= h_1 \underline{e}_1 du_1 + h_2 \underline{e}_2 du_2 + h_3 \underline{e}_3 du_3 .$$

Therefore:

$$\frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3 = f_1 h_1 du_1 + f_2 h_2 du_2 + f_3 h_3 du_3 \quad -(20)$$

so

$$f_i = \frac{1}{h_i} \frac{\partial f}{\partial u_i} \quad -(21)$$

$i = 1, 2, 3.$

so:

$$\underline{\nabla} f = \frac{\underline{e}_1}{h_1} \frac{\partial f}{\partial u_1} + \frac{\underline{e}_2}{h_2} \frac{\partial f}{\partial u_2} + \frac{\underline{e}_3}{h_3} \frac{\partial f}{\partial u_3} \quad -(22)$$

giving eq. (8).

The scale factors appear in the denominators.

The above is a completely general formalism and can be illustrated with the cylindrical polar coordinates.

c)

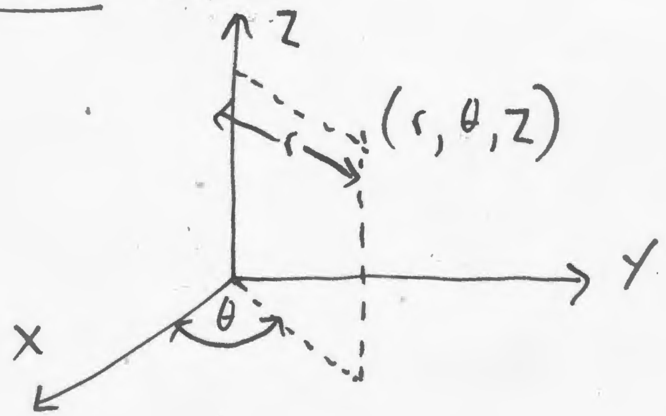
Cylindrical Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2 + y^2 = r^2$



Therefore:

$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + z\,\underline{k} \quad -(23)$$

$$\frac{\partial \underline{r}}{\partial r} = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(24)$$

$$\frac{\partial \underline{r}}{\partial \theta} = -r\sin\theta\,\underline{i} + r\cos\theta\,\underline{j} \quad -(25)$$

$$\frac{\partial \underline{r}}{\partial z} = \underline{k} \quad -(26)$$

The scale factors are:

$$h_1 = h_r = \left|\frac{\partial \underline{r}}{\partial r}\right| = \left(\cos^2\theta + \sin^2\theta\right)^{1/2} = 1 \quad -(27)$$

$$h_2 = h_\theta = \left|\frac{\partial \underline{r}}{\partial \theta}\right| = r \quad -(28)$$

$$h_3 = h_z = \left|\frac{\partial \underline{r}}{\partial z}\right| = 1 \quad -(29)$$

The unit vectors are:

$$\underline{e}_r = \underline{e}_1 = \frac{1}{h_1}\frac{\partial \underline{r}}{\partial r} = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(30)$$

$$\underline{e}_\theta = \underline{e}_2 = \frac{1}{h_2}\frac{\partial \underline{r}}{\partial \theta} = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(31)$$

$$\underline{e}_z = \underline{e}_3 = \frac{1}{h_3}\frac{\partial \underline{r}}{\partial z} = \underline{k} \quad -(32)$$

) The $\underline{\nabla}$ is:

$$\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{\partial f}{\partial z}\underline{e}_z \quad -(33)$$

Note that the components have the correct units.

In eq. (9):

$$d\underline{r}\cdot d\underline{r} = dr^2 + r^2 d\theta^2 + dZ^2 \quad -(34)$$

In eq. (14):

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - dZ^2 \quad -(35)$$

In the plane:

$$dZ^2 = 0 \quad -(36)$$

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(37)$$

In the plane:

$$du_3^2 = 0 \quad -(38)$$

$$\boxed{ds^2 = c^2 dt^2 - h_1^2 du_1^2 - h_2^2 du_2^2} \quad -(39)$$

Orbital Constraints

The orbit is defined by:

$$\frac{dr}{d\theta} = f(\theta) \quad -(40)$$

where

$$f = f(r, \theta, t) \quad -(41)$$

In general:

$$\boxed{\frac{du_1}{du_2} = f(u_1, u_2, t)} \quad -(42)$$

From eq. (42) in eq. (41):

$$ds^2 = c^2 dt^2 - h_1^2 du_1^2 - \frac{h_2^2}{f^2} du_1^2 \quad -(43)$$

i.e

$$\boxed{ds^2 = c^2 dt^2 - \left(h_1^2 + \frac{h_2^2}{f^2}\right) du_1^2} \quad -(44)$$

From eq. (44):

$$\boxed{g_{00} = 1, \quad g_{11} = -\left(h_1^2 + \frac{h_2^2}{f^2}\right)} \quad -(45)$$

which gives the two metric elements in the general curvilinear coordinate.

The connections are found from metric compatibility:

$$D_\rho g_{\mu\nu} = 0 \quad -(46)$$

giving the torsion and curvature elements.

7) The antisymmetric connection is given by:

$$\Gamma^{\alpha}_{\rho\alpha} = \frac{1}{2} g^{\alpha\alpha} \partial_\rho g_{\alpha\alpha} . \quad -(47)$$

In a plane there are two connections:

$$\boxed{\Gamma^{1}_{01} = \frac{1}{c} \frac{df / dt}{2(1+f)}} \quad -(48)$$

and

$$\Gamma^{1}_{21} = \frac{1}{2} g^{11} \partial_2 g_{11} \quad -(49)$$

$$= \frac{\partial_2 g_{11}}{2 g_{11}}$$

where

$$\boxed{\partial_2 = \frac{1}{h_2} \frac{\partial}{\partial u_2}} \quad -(50)$$

In cylindrical polar coordinates:

$$\boxed{\partial_2 = \frac{1}{r} \frac{\partial}{\partial \theta}} \quad -(51)$$

Therefore

$$\boxed{\Gamma^{1}_{21} = \frac{df / d\theta}{2r(1+f)}} \quad -(52)$$

The two Evans identities are:

$$D_0 T^{1}_{01} := R^{1}_{001} \quad -(53)$$

$$D_2 T^{1}_{21} := R^{1}_{221} \quad -(54)$$

8).

In general curvilinear coordinates these reduce to:

$$6(1+f)\frac{d^2 f}{dt^2} = 5\left(\frac{df}{dt}\right)^2 \quad -(55)$$

$$6(1+f)\frac{d^2 f}{du_2^2} = 5\left(\frac{df}{\partial u_2}\right)^2 \quad -(56)$$

i.e. $$\frac{d^2 f}{dt^2} = \left(\frac{du_2}{dt}\right)^2 \frac{d^2 f}{\partial u_2^2} \quad -(57)$$

which as in note 205(4) is the result of the chain rule.

Torsions in Cylindrical Polar Coordinates

$$T^1{}_{01} = \frac{1}{c}\frac{\partial f/\partial t}{1+f} \quad -(58)$$

$$T^1{}_{21} = \frac{1}{r}\frac{\partial f/\partial \theta}{1+f} \quad -(59)$$

Curvatures in Cylindrical Polar Coordinates

These are given by computer algebra. The elements that appear in ~~the~~ Evans identity are:

$$R^1{}_{001} = \frac{1}{c^2}\left(\frac{2(1+f)\,d^2 f/dt^2 - (df/dt)^2}{4(1+f)^2}\right) \quad -(60)$$

$$R^1{}_{221} = \frac{1}{r^2}\left(\frac{2(1+f)\,d^2 f/d\theta^2 - (df/d\theta)^2}{4(1+f)^2}\right) \quad -(61)$$